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WORKS BY A. E. PIERPOINT, B.Sc.

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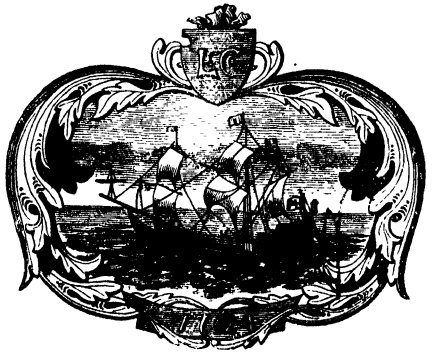

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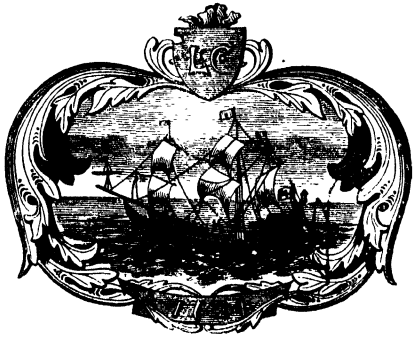

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PREFACE

In completing this book, I have arranged the material on the same plan as in Part I.

The Miscellaneous Examples at the end are graduated, and designed to cover the whole subject. Some of them are original, but the greater number have been selected from examination papers set within recent years by the various Indian Universities and Engineering Colleges. There are three hundred of them, and they bring the total number of examples to something over two thousand.

Every example has been worked out several times, so that I venture to hope that few mistakes will be found among the answers.

The student is advised to omit the Examination Questions at the ends of chapters when reading through the book for the first time.

My warm thanks are due to my colleague, Mr. W. H. G. Padfield, M.A., for his careful revision of the proof-sheets; and I also desire to acknowledge the help that I have received from many who have used Part I., and who have given me the benefit of their advice.

A. E. PIERPOINT.

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MENSURATION

PART II.

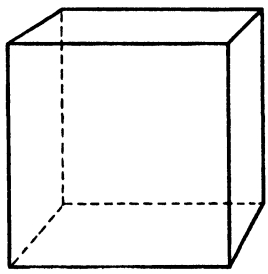
CHAPTER XIX.

INTRODUCTORY—TABLE OF UNITS.

108. A *solid figure* or *solid* is bounded by one or more surfaces. Hence a solid must have length, breadth (width), and thickness (depth or height).

When *plane* surfaces bound a solid they are called its *faces*, and the solid is called a *polyhedron*. The lines which bound the faces of a solid are called its *edges*.[1] The *volume* of a solid is the amount of space enclosed by its bounding surface or surfaces.

109. When the bounding surfaces of a solid are six equal squares, the solid is called a *cube*. It is evident that the length, breadth, and thickness of a cube are equal to one another (see figure).



A cube is called a *cubic inch*, a *cubic foot*, or a *cubic yard* according as its length, breadth, and thickness are each a linear inch, a linear foot, or a linear yard.

110. The units of volume most commonly used in mensuration are given in the following table:—

Cubic or Solid Measure.

1728 cubic inches make 1 cubic foot.
27 cubic feet make 1 cubic yard.

111. One cubic inch of pure water weighs 252·458 grains (Troy).

[1] Some writers do not limit the term *faces* to the *plane* surfaces that bound a solid. All the bounding surfaces of a solid they call its *faces*, whether plane or curved, and the lines of intersection of adjacent faces they call its *edges*.

$\therefore$ one cubic foot of pure water weighs $252{\cdot}458 \times 1728$ grs. (Troy)

$$= \frac{252{\cdot}458 \times 1728 \times 16}{7000} \text{ ozs. (Av.)}$$

$$= 997{\cdot}137 \text{ ozs. (Av.)}$$

This weight differs so little from 1000 ozs. that in practice it is usual to take 1000 ozs., or $62\frac{1}{2}$ lbs., as the weight of one cubic foot of pure water. This assumption is always made in the examples that follow unless otherwise stated.

112. "*A pint of pure water*
Weighs a pound and a quarter;"

$\therefore$ a pint of pure water weighs $1\frac{1}{4} \times 7000$ grs. (Troy)

$\therefore$ a pint measure contains $\frac{1\frac{1}{4} \times 7000}{252{\cdot}458}$ cub. in.

$\therefore$ a gallon measure contains $\frac{1\frac{1}{4} \times 7000 \times 8}{252{\cdot}458}$ cub. in.

$$= 277{\cdot}274 \text{ cub. in.}$$

This volume differs so little from $277\frac{1}{4}$ cub. in. that in practice it is usual to take $277\frac{1}{4}$ cub. in. as the capacity of one gallon. This assumption is always made in the examples that follow unless otherwise stated.

ILLUSTRATIVE EXAMPLES.

113. *Example* 1.—How many cubic inches are there in 2 cub. yds. 16 cub. ft. 1044 cub. in.?

2 cub. yds. 16 cub. ft. 1044 cub. in.

27

70 cub. ft.

1728

120960

1044

122004 cub. in.

$\therefore$ 2 cub. yds. 16 cub. ft. 1044 cub. in. = 122004 cub. in.

Example 2.—Reduce 526072 cub. in. to cubic yards, etc.

144 { 12)526072 cub. in.
12)43839 − 4
12)3653 − 3 } 40 } 760 cub. in.
9)304 − 5
3)33 − 7 } 7 cub. ft.
11 − 0

$\therefore$ 526072 cub. in. = 11 cub. yds. 7 cub. ft. 760 cub. in.

Example 3.—What fraction of a pint has the capacity of 1 cub. in. ?

$277\frac{1}{4}$ cub. in. is the capacity of 1 gallon . . . . . § 112.

$\therefore$ 1 „ „ $\frac{1 \times 8}{277\frac{1}{4}}$ of a pint

$= \frac{8 \times 4}{1109}$ of a pint

$= \frac{32}{1109}$ of a pint

Examples – XIX.

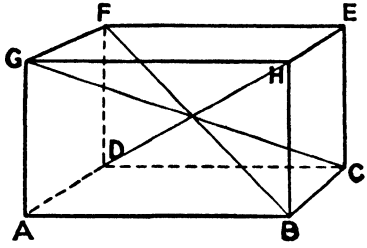
1. Reduce 10 cub. ft. 736 cub. in. to cubic inches.
2. Reduce 2 cub. yds. 18 cub. ft. 1232 cub. in. to cubic inches.
3. How many cubic yards, cubic feet, cubic inches are there in 230,000 cub. in. ?
4. Reduce 136,592 cub. in. to cubic yards, cubic feet, cubic inches.
5. Find in cubic feet the space occupied by a ton of pure water.
6. Find the weight of pure water that occupies a space of 1 cub. yd.
7. Find the cubical content of a quart jug.
8. Find the weight of 50 gallons of pure water.

CHAPTER XX.

ON RECTANGULAR SOLIDS.

114. A *rectangular solid* is a figure bounded by six rectangular faces. It follows, from this definition, that the opposite faces of a rectangular solid are equal rectangles lying in parallel planes.

Thus in the rectangular solid AFH, the rectangle $ABCD$ is equal to the rectangle $EFGH$, and the two rectangles lie in parallel planes. A common brick may be taken as a familiar example of a rectangular solid.



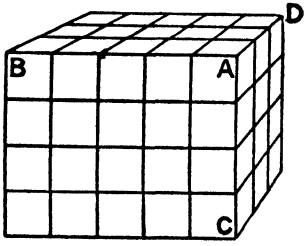
The length, breadth, and depth of a rectangular solid are called its *dimensions*.

The four straight lines joining the opposite corners of a rectangular solid are called its *diagonals*. For example, DH, GC, FB are three of the diagonals of the rectangular solid AFH.

When the dimensions of a rectangular solid are equal to one another, the figure is called a *cube* (§ 109). When the dimensions of a rectangular solid are not equal to one another, the figure is called a *cuboid*. A rectangular solid is sometimes called a *parallelopipedon*.

PROPOSITION XXVIII.

115. *To find the volume of a rectangular solid, having given its dimensions.*

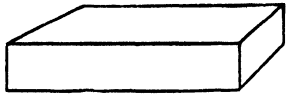


Let $ABCD$ be a rectangular solid such that AB represents a length of 5 in., AC a length of 4 in., and AD a length of 3 in.

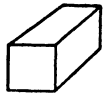
It is required to find the volume of the rectangular solid $ABCD$.

Divide AB into five equal parts, AC into four equal parts, and AD into three equal parts. Then each of these equal parts will represent an inch.

Through the points of division in AC draw planes parallel to the face ABD. These will divide the solid into four equal layers, each layer being 1 in. thick (see figure).



Again, through the points of division in AB draw planes parallel to the face ADC. These will divide each layer into five equal bars, each bar measuring 1 sq. in. in cross-section (see figure).



Again, through the points of division in AD draw planes parallel to the face ABC. These will divide each bar into three equal blocks, each block being a cubic inch (see figure).

Note that the number of layers is the same as the number of inches in AC, the number of bars in each layer is the same as the number of inches in AB, and the number of cubic inches in each bar is the same as the number of inches in AD.

Therefore the rectangular solid contains—

$$4 \times 5 \times 3 \text{ cub. in.} = 60 \text{ cub. in.}$$

From this special case we may arrive at the general conclusion—

If the length, breadth, and depth of a rectangular solid measure a, b, and c *of the same linear unit* respectively, then the volume of the solid will measure $a \times b \times c$ of the *corresponding solid unit.*

Hence rule—

The continued product of the numbers of any linear unit in the dimensions of a rectangular solid will give the number of the corresponding solid unit in the volume.

Or briefly—

$$\begin{aligned} \textbf{Volume of rectangular solid} &= \textbf{length} \times \textbf{breadth} \times \textbf{depth} \\ &= \textbf{length} \times \textbf{area of end} \\ &= \textbf{breadth} \times \textbf{area of side} \\ &= \textbf{depth} \times \textbf{area of base} \\ \mathbf{V} &= \mathbf{a} \times \mathbf{b} \times \mathbf{c} \quad . \quad . \quad . \quad \text{(i.)} \end{aligned}$$

Hence—

$$\left.\begin{matrix}\textbf{Length of rectangular} \\ \textbf{solid}\end{matrix}\right\} = \frac{\textbf{volume}}{\textbf{breadth} \times \textbf{depth}} = \frac{\textbf{volume}}{\textbf{area of end}}$$

$$\mathbf{a} = \frac{\mathbf{V}}{\mathbf{b} \times \mathbf{c}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

and—

$$\left.\begin{matrix}\textbf{Breadth of rectangular}\\ \textbf{solid}\end{matrix}\right\} = \frac{\textbf{volume}}{\textbf{length} \times \textbf{depth}} = \frac{\textbf{volume}}{\textbf{area of side}}$$

$$b = \frac{V}{a \times c} \quad \ldots\ldots\ldots \text{(iii.)}$$

and

$$\left.\begin{matrix}\textbf{Depth of rectangular}\\ \textbf{solid}\end{matrix}\right\} = \frac{\textbf{volume}}{\textbf{length} \times \textbf{breadth}} = \frac{\textbf{volume}}{\textbf{area of base}}$$

$$c = \frac{V}{a \times b} \quad \ldots\ldots\ldots \text{(iv.)}$$

116. The volume of a rectangular solid can be expressed in terms of the areas of its base, a side, and an end, thus—

Volume of rectangular solid

$$= \sqrt{(\textit{length} \times \textit{breadth}) \times (\textit{length} \times \textit{depth}) \times (\textit{breadth} \times \textit{depth})}$$

$$= \sqrt{(\textit{area of base}) \times (\textit{area of side}) \times (\textit{area of end})}$$

$$V = \sqrt{A_1 \times A_2 \times A_3}$$

PARTICULAR CASE.

117. Cube.

Here the dimensions are equal to one another.

That is, length = breadth = depth = edge

$$\left.\begin{matrix}\text{and volume of any}\\ \text{rectangular solid}\end{matrix}\right\} = \text{length} \times \text{breadth} \times \text{depth} \quad \S\ 115.$$

$$\therefore \text{volume of cube} = (\text{edge})^3$$

$$V = a^3 \quad \ldots\ldots\ldots \text{(i.)}$$

$$\therefore \text{edge of cube} = \sqrt[3]{\text{volume}}$$

$$a = \sqrt[3]{V} \quad \ldots\ldots\ldots \text{(ii.)}$$

Hence rule—

The cube root of the number of any solid unit in the volume of a cube gives the number of the corresponding linear unit in the edge.

Or briefly—

$$\textbf{Edge of cube} = \sqrt[3]{\textbf{volume}}$$

$$a = \sqrt[3]{V}$$

PROPOSITION XXIX.

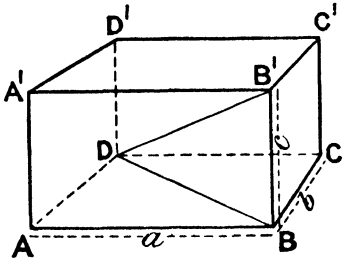
118. *To find a diagonal of a rectangular solid, having given its dimensions.*

Let $ABC'D'$ be a rectangular solid. Let its dimensions BA, BC, BB' measure a, b, c of the same linear unit respectively.

It is required to find the length of the diagonal DB' in terms of a, b, c.

Join DB.

Because DB' is the hypotenuse of a right-angled triangle DBB',



$$\therefore DB'^2 = DB^2 + BB'^2 \quad \text{Euc. I. 47.}$$

$$\text{But } DB^2 = AB^2 + AD^2 \quad \text{Euc. I. 47.}$$

$$= AB^2 + BC^2 \quad \text{Euc. I. 34.}$$

$$\therefore DB'^2 = AB^2 + BC^2 + BB'^2$$

$$\therefore \text{square on } DB' = (a^2 + b^2 + c^2) \text{ square units} \quad \S\ 9.$$

$$\therefore DB' = \sqrt{a^2 + b^2 + c^2} \text{ linear units} \quad \S\ 9.$$

Hence rule—

Add together the squares of the numbers of any linear unit in the dimensions of a rectangular solid; then the square root of the sum will give the number of the same linear unit in the diagonal.

Or briefly—

$$\left.\begin{matrix}\textbf{Diagonal of rectangular solid}\end{matrix}\right\} = \sqrt{(\textbf{length})^2 + (\textbf{breadth})^2 + (\textbf{depth})^2}$$

$$\mathbf{d = \sqrt{a^2 + b^2 + c^2}}$$

Note.—It is obvious that we shall obtain the same expression for each of the four diagonals of the rectangular solid $ABC'D'$. Hence we infer that *all the diagonals of a rectangular solid are equal to one another.*

PARTICULAR CASE.

119. Cube.

Here length = breadth = depth = edge

$$\left.\begin{matrix}\text{and diagonal of any}\\ \text{rectangular solid}\end{matrix}\right\} = \sqrt{(\text{length})^2 + (\text{breadth})^2 + (\text{depth})^2} \quad \S\ 118.$$

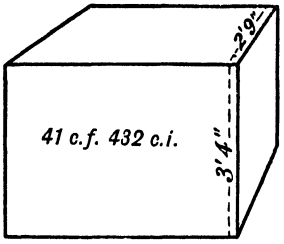
$$\therefore \text{diagonal of cube} = \sqrt{3 \times (\text{edge})^2}$$

$$d = \sqrt{3a^2}$$

$$= a\sqrt{3} \quad \ldots\ldots\ldots\ldots \text{(i.)}$$

$$\text{hence } a = \frac{d}{\sqrt{3}} \quad \ldots\ldots\ldots\ldots \text{(ii.)}$$

ILLUSTRATIVE EXAMPLES.



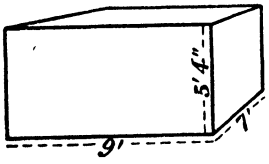
120. *Example* 1.—Find the length of a rectangular solid whose volume measures 41 cub. ft. 432 cub. in., breadth 2 ft. 9 in., and depth 3 ft. 4 in.

$$\text{Length of rectangular solid} = \frac{V}{b \times c} \text{ ft.} \quad \S\ 115.$$

where $V = 41\frac{432}{1728} = 41\frac{1}{4}$,
and $b = 2\frac{3}{4}$,
and $c = 3\frac{1}{3}$;

$$\therefore \text{length of rectangular solid} = \frac{41\frac{1}{4}}{2\frac{3}{4} \times 3\frac{1}{3}} \text{ ft.}$$
$$= \frac{165 \times 4 \times 3}{4 \times 11 \times 10} \text{ ft.}$$
$$= 4\tfrac{1}{2} \text{ ft.}$$
$$= 4 \text{ ft. } 6 \text{ in.}$$

Example 2.—How many tons of water will a rectangular cistern hold whose length is 9 ft., breadth 7 ft., and depth 5 ft. 4 in.?



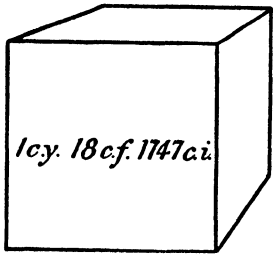
$$\text{Volume of cistern} = (a \times b \times c) \text{ cub. ft.} \quad \S\ 115.$$

where $a = 9$,
$b = 7$,
$c = 5\frac{1}{3}$;

$\therefore$ volume of cistern $= (9 \times 7 \times 5\frac{1}{3})$ cub. ft.

And 1 cub. ft. of water weighs 1000 ozs. (Av.) . . . . § 111

$\therefore$ weight of water in cistern $= 9 \times 7 \times 5\frac{1}{3} \times 1000$ ozs.
$$= \frac{9 \times 7 \times 16 \times 1000}{3 \times 16 \times 28 \times 4 \times 20} \text{ tons}$$
$$= 9\tfrac{3}{8} \text{ tons}$$



Example 3.—Find the edge of a cube whose volume measures 1 cub. yd. 18 cub. ft. 1747 cub. in.

$$\text{Edge of cube} = \sqrt[3]{V} \text{ in.} \quad \S\ 117.$$

where $V = 79507$;

$$\therefore \text{edge of cube} = \sqrt[3]{79507} \text{ in.}$$
$$= 43 \text{ in.}$$
$$= 1 \text{ yd. } 7 \text{ in.}$$

Example 4.—How many maunds of kankar will be required to metal one mile of road 18 ft. wide, and an average of 9 in. thick, the kankar weighing $1\frac{1}{2}$ maunds per cubic foot?

Volume of kankar required = $(a \times b \times c)$ cub. ft. . § 115.

where $a = 1760 \times 3$,
$b = 18$,
$c = \frac{3}{4}$;

$\therefore$ volume of kankar required $= \frac{1760 \times 3 \times 18 \times 3}{4}$ cub. ft.

$\therefore$ weight „ „ $= \frac{1760 \times 3 \times 18 \times 3 \times 3}{4 \times 2}$ maunds

$= 106{,}920$ maunds

Example 5.—Three cubes of metal, whose edges are 3, 4, and 5 in. respectively, are melted down and formed into a single cube. Find its diagonal.

Volume of the single cube $= (3^3 + 4^3 + 5^3)$ cub. in. . § 117.
$= 216$ cub. in.

$\therefore$ edge „ „ $= \sqrt[3]{216}$ in. . . . . § 117.
$= 6$ in.

$\therefore$ diagonal „ „ $= 6\sqrt{3}$ in. . . . . § 119.
$= 10{\cdot}392$ in.

Example 6.—If a brick occupy a space of 9 in. by $4\frac{1}{2}$ by 3, how many would be required for a wall 100 ft. in length, 10 ft. in height, and a brick and a half in thickness?

Number of bricks required $= \frac{\text{volume of wall}}{\text{volume of a brick}}$

$= \frac{(100 \times 10 \times \frac{27}{2^4})\text{ cub. ft.}}{(\frac{1}{1728} \times 9 \times \frac{9}{2} \times 3)\text{ cub. ft.}}$. § 115.

$= \frac{100 \times 10 \times 27 \times 1728 \times 2}{24 \times 9 \times 9 \times 3}$

$= 16{,}000$

Example 7.—If gold be beaten out so thin that an ounce (Troy) will form a leaf of 20 sq. yds., find how many of these leaves will make an inch thick, the weight of a cubic foot of gold being 10 cwt. 95 lbs.

Because 10 cwt. 95 lbs. is the weight of 1 cub. ft. of gold

$\therefore$ 1 oz. is the weight of $\frac{1728}{1215 \times 12}$ cub. in. of gold

Hence, if x in. be the thickness of a leaf—

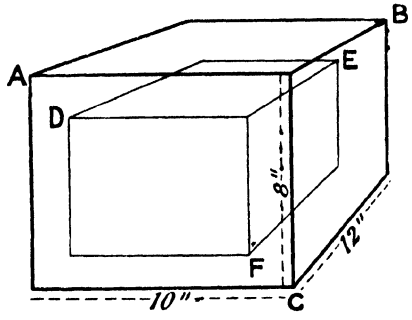
$20 \times 9 \times 144 \times x = \frac{1728}{1215 \times 12}$ . . . . . § 115.

$\therefore x = \frac{1728}{1215 \times 12 \times 20 \times 9 \times 144}$

$= \frac{1}{218700}$

$\therefore$ 218,700 leaves will make an inch thick

Example 8.—A box with a lid measures 12 in. long, 10 broad, and 8 deep, outside measure. It is made of wood 2 in. thick. Required its weight empty, the specific gravity of the wood being 0·75.



Volume of wood = volume of solid ABC − volume of solid DEF
= (12 × 10 × 8) cub. in. − (8 × 6 × 4) cub. in. § 115.
= 768 cub. in.

$$\left.\begin{array}{r}\text{Now, specific gravity}\\ \text{of the wood}\end{array}\right\} = \frac{\text{weight of any volume of the wood}}{\text{weight of an equal volume of water}}$$

$$\left.\begin{array}{r}\text{and weight of 768 cub.}\\ \text{in. of water}\end{array}\right\} = \frac{1000 \times 768}{1728} \text{ ozs.} \quad \ldots \quad \S\,111.$$

$$\therefore 0{\cdot}75 = \frac{\text{weight of the wood} \times 1728}{1000 \times 768 \text{ ozs.}}$$

$$\therefore \text{weight of the wood} = \frac{3 \times 1000 \times 768}{4 \times 1728} \text{ ozs.}$$

$$= \frac{1000}{3} \text{ ozs.}$$

$$= 20 \text{ lbs. } 13\tfrac{1}{3} \text{ ozs.}$$

Examples—XX.

Find the volumes of rectangular solids having the following dimensions:—

1. Length 7 ft., breadth 6 ft., height 5 ft.

2. Length 5 ft. 6 in., breadth 4 ft. 9 in., depth 4 ft. 3 in.

3. Length 2 yds. 2 ft. 9 in., breadth 1 yd. 2 ft., depth 1 yd. 1 ft. 6 in.

4. Find the length of a rectangular solid whose volume measures 18 cub. ft. 390 cub. in., width 2 ft. 5 in., and depth 3 in.

5. Find the width of a rectangular solid whose volume measures 56 cub. ft. 1440 cub. in., length 16 ft. 6 in., and depth 8 in.

6. Find the depth of a rectangular solid whose volume is 14 cub. ft. 1008 cub. in., length 5 ft. 10 in., and breadth 3 ft. 4 in.

7. Find the area of the base of a rectangular solid whose volume is 23 cub. ft., 396 cub. in., and depth 2 ft. 6 in.

8. Find the depth of a rectangular solid whose volume is 9 cub. yds. 4 cub. ft. 1584 cub. in., and base 6 sq. yds. 4 sq. ft. 48 sq. in.

9. Find the area of each end of a rectangular solid whose volume is 4 cub. yds. 6 cub. ft. 1008 cub. in., and length 5 ft.

10. Find the volume of a rectangular solid whose base is 3 sq. yds. 8 sq. ft. 28 sq. in., and height 1 yd. 2 ft. 10 in.

11. Find the length of a rectangular solid whose volume is 4 cub. yds. 23 cub. ft. 432 cub. in., and whose ends each measure 2 sq. yds. 4 sq. ft. 72 sq. in.

12. Find the volume of a rectangular solid whose sides each measure 3 sq. yds. 5 sq. ft. 68 sq. in., and whose width is 1 yd. 2 ft. 8 in.

13. Find the cost of a rectangular solid whose length, breadth, and depth are 10 in., 8 in., and 7 in., at the rate of Rs.3 8 annas per cubic inch.

14. How many bricks will be required to build a wall 80 ft. long, 18 in. thick, and 6 ft. high, a brick being 9 in. long, $4\frac{1}{2}$ in. wide, and 3 in. deep?

15. Find to the nearest gallon the capacity of a cistern having the following dimensions: length 7 ft. 8 in., breadth 7 ft. 2 in., depth 6 ft. 8 in.

16. Find the weight of a rectangular block of wood 4 ft. 3 in. long, 2 ft. 9 in. wide, 2 ft. 6 in. deep, at 30 lbs. per cubic foot.

17. What weight of water will a rectangular cistern hold whose dimensions are 10 ft., 8 ft., and 7 ft.? Give the result in tons, cwts., etc.

18. The length and width of a rectangular tank are 3 ft. 6 in. and 2 ft. 9 in., and the tank is capable of holding 56 gallons of water: find its depth to the nearest inch.

19. If the length and width of a tank are 18 ft. 8 in. and 14 ft. 6 in., what volume of water must be drawn off to make the surface sink 1 ft.?

20. A box without a lid is made of wood an inch thick: find the cubical content of the box if its external dimensions are 2 ft. 8 in., 2 ft. 4 in., and 1 foot 9 in.

Find the volumes of cubes having the following edges:—

21. 2 ft. 10 in.

22. 1 yd. 2 ft. 9 in.

23. 2 poles.

24. 1 pole 2 yds.

Find the edges of cubes having the following volumes:—

25. 3 cub. ft. 1675 cub. in.

26. 1 cub. yd. 19 cub. ft. 19 cub. in.

27. 4 cub. yds. 23 cub. ft. 613 cub. in.

28. 163 cub. yds. 23 cub. ft. 701 cub. in.

29. The base of a rectangular cistern measures 18 sq. ft.: find its depth to the nearest inch if it is capable of holding 500 gallons of water.

30. Find to the hundredth part of an inch the edge of a cubical vessel whose capacity is 50 gallons.

31. How many square inches of surface can be covered uniformly by 1 cub. ft. of gold, if the coating is 0·000005 in. thick?

32. Find the cubical content of a box without a lid which is made of wood an inch thick, if its external length, breadth, and depth are 2 ft. 2 in., 1 ft. 10 in., and 1 ft. 9 in. respectively.

33. How many cubic inches of wood are required to build a box without a lid if the wood is half an inch thick, and if the external length, breadth, and depth of the box are 2 ft. 9 in., 2 ft. 3 in., and 1 ft. 10 in. respectively?

34. How many cubic inches of wood are required to build a box with a lid, if the wood is 1 in. thick, and if the internal dimensions of the box are 1 ft. 5 in., 1 ft. 3 in., and 1 ft. 1 in.?

35. If 300 gallons of water are poured into a rectangular cistern 12 ft. 9 in. long and 10 ft. 6 in. wide, through how many inches will the surface rise?

36. If each of the dimensions of any rectangular solid be doubled, show that its volume will be increased eightfold.

37. If the internal dimensions of a rectangular cistern are 3 ft. 6 in., 2 ft. 10 in., and 1 ft. 8 in., find to the nearest second how long it will take a pipe to fill it, if the pipe admits 8 gallons a minute.

38. The dimensions of a rectangular solid are in the proportion of the numbers 2, 5, and 7, and its volume is 13 cub. ft. 1546 cub. in. : find its dimensions.

39. Find the volume of a cube whose diagonal measures 2 ft. 6 in.

40. Find the diagonal of a rectangular solid whose length, breadth, and depth measure 3 ft. 4 in., 2 ft. 9 in., and 2 ft. 3 in. respectively.

41. Find the diagonal of a cube whose volume measures 10 cub. ft. 296 cub. in.

42. Find the depth of a rectangular solid whose diagonal, length, and breadth measure 7 ft. 3 in., 5 ft. 3 in., and 3 ft. respectively.

43. The volume of a rectangular solid is 2160 cub. ft., and its diagonal is 25 ft. If the length is 20 ft., find the breadth and depth.

44. Find the edge of a cube equal in volume to a rectangular solid whose dimensions are 3 ft. 9 in., 1 ft. 3 in., and 5 in.

Examination Questions—XX.

A. *Allahabad University ; Intermediate.*

1. It is desired to put a cubical case whose content is 4019·679 cub. ft. through a square hatchway whose area is 37,791·36 sq. in. : show whether this can be done.

B. *Punjab University ; First Exam. in Civil Engineering.*

2. Three cubes of metal whose edges are 3, 4, and 5 in. respectively, are melted and formed into a single cube : if there be no waste in the process, show that the edge of the new cube will be 6 in.

3. Find the length of the longest rod that can be placed in a room 30 ft. long, 24 ft. broad, and 18 ft. high.

C. *Madras University : B.E. Exam.*

4. Water is distributed to a town of 50,000 inhabitants from a reservoir consisting of three compartments, 200 ft. × 100 ft., with vertical sides, and 12 ft. depth of water. The allowance is 15 gallons per head per day. How many days' supply will the reservoir hold ?

D. *Calcutta University ; F.E. Exam.*

5. In measuring the edges of a cubical box to ascertain its content, an error of 0·202 in. is made in excess for the length, and of 0·2 in. in defect for the breadth, the height being properly measured. The calculated volume agrees with the true volume. Find the volume in cubic inches.

E. *Sibpur Apprentice Dept. : Monthly Exam.*

6. A box without a lid is made of wood an inch thick ; the external length, breadth, and height of the box are 2 ft. 10 in., 2 ft. 5 in., and 1 ft. 7 in. respectively : find what volume the box will hold, and the number of cubic inches of wood.

7. The external length, breadth, and height of a closed rectangular wooden box are 18 in., 10 in., and 6 in. respectively, and the thickness of the wood is half an inch. When the box is empty it weighs 15 lbs., and when filled with sand 100 lbs. Find the weight of a cubic inch of wood and of a cubic inch of sand.

8. A reservoir is 24 ft. 8 in. long by 12 ft. 9 in. wide : find how many cubic feet of water must be drawn off to make the surface sink 1 ft.

F. *Sibpur Apprentice Dept. : Annual Exam.*

9. The diagonal of a cube is 30 in. : what is the solid content ?

G. *Sibpur Apprentice Depart. : Final Exam.*

10. A schoolroom is to be built to accommodate 70 children, so as to allow $8\frac{1}{2}$ sq. ft. of floor and $110\frac{1}{8}$ cub. ft. of space for each child : if the room be 34 ft. long, what must be its breadth and height ?

H. *Roorkee Engineer : Entrance.*

11. Taking the dimensions of a brick to be $9'' \times 4\frac{1}{2}'' \times 3''$, find the number required to build a storeroom 14 ft. high and $22' \times 15'$, the walls being 2 ft. thick and the room being provided with a doorway $8' \times 3\frac{1}{2}'$ and two windows $3' \times 2'$.

12. Find how many bricks, of which the length, breadth, and thickness are 9, $4\frac{1}{2}$, and 3 in., will be required to build a wall of which the length, height, and thickness are 72, 8, and $1\frac{1}{2}$ ft.

13. The three conterminous edges of a rectangular solid are 36, 75, and 80 in. respectively : find the edge of a cube which will be of the same capacity.

14. A river 25 ft. deep and 480 ft. wide is flowing at the rate of 3 miles an hour : find how many tons of water run into the sea per minute.

15. A cubic foot of gold is extended by hammering so as to cover an area of six acres : find the thickness of the gold in decimals of an inch, correct to the first two significant figures.

I. *Roorkee Upper Subordinate : Entrance.*

16. A rectangular bath is 14 ft. long, 9 ft. wide, and 4 ft. deep : how much deeper must it be made to hold 180 gallons more ?

17. A reservoir contains 3,217,428 cub. ft. of water ; its depth is one-third of its length, and its breadth is half the difference between the length and one-third of the depth : find the dimensions.

18. A box without a lid measures externally 4 ft. long, 3 ft. wide, and 2 ft. deep ; the material has a uniform thickness of $\frac{3}{4}$ in. If the wood cost 7*s.* 9*d.* per cubic foot, and the making $\frac{1}{10}$ of the material, find to the nearest penny the cost of the box.

19. A rectangular solid is 13 ft. long, $3\frac{1}{2}$ ft. broad, and 2 ft. high : find the length of its diagonal, and also the area of a plane passing through two opposite edges of $3\frac{1}{2}$ ft.

Additional Examination Questions—XX.

20. Find the least length of wall that will be required to enclose a space of 1200 sq. yds. by the side of an existing wall, so that only three sides require to be walled up. If the average sectional area of the wall be 18 sq. ft., find the cost of building it with stones 18 in. × 9 in. × $4\frac{1}{2}$ in. at Rs. 80 per 1000 stones. (Madras University : B.E. Exam.)

21. A rectangular reservoir is 100 ft. long by 64 ft. broad : at what rate of speed per hour must water flow into it through a pipe whose cross-section is a square of side 2 in., in order to make the water rise 2 ft. in 8 hrs. ? (Bombay University : L.C.E. Exam.)

22. If a be the length of each edge of a cube, show that the diagonal of each face is $a\sqrt{2}$, and the diagonal of the solid $a\sqrt{3}$. (Roorkee Upper Subordinate : Monthly.)

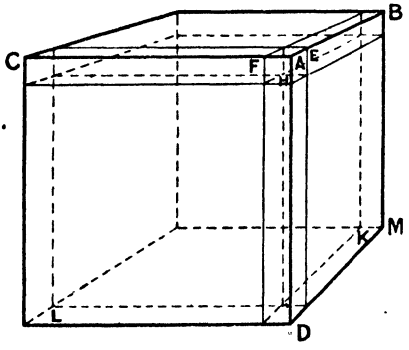
23. How many superficial feet of inch plank can be sawn out of a log of timber 20 ft. 7 in. long, 1 ft. 10 in. wide, 1 ft. 8 in. deep ? (Roorkee Engineer : Final.)

24. How many sovereigns will a box 16 in. long, 7 in. deep, and 5 in. broad, contain if each sovereign measure $\frac{7}{8}$ in. × $\frac{1}{16}$ in. ? (Roorkee Upper Subordinate : Monthly.)

CHAPTER XXI.

ON DUODECIMALS—(continued).

121. We have seen (Chapter III.) that the *area* of a rectangle can conveniently be found by using duodecimals. It is now proposed to extend this method to the determination of the *volumes* of rectangular solids.



122. Consider the cube *ABCD*.

Let each of its edges be taken to represent a length of 1 ft. 1 in. On the same scale, from *B*, from *C*, and from *D*, along *BA* and *CA* and *DA* respectively, divide off a length corresponding to 1 ft.

Then the remaining parts of these three lines will be lengths each corresponding to 1 in.

Through the points of division draw planes parallel to the sides of the cube.

We now see that the whole cube is made up of several pieces of four different sizes—

The largest size is a cube, *HKL*, measuring 1 ft. each way, that is, a cubic foot.

The second largest size is a rectangular solid, *MHK*, measuring 1 ft. by 1 ft. by 1 in. This is evidently the twelfth part of a cubic foot, and we have already spoken of it as a cubic or solid prime (§ 13). Of these there are three.

The third largest size is a rectangular solid, *CFH*, measuring 1 ft. by 1 in. by 1 in. This is evidently the twelfth part of a cubic or solid prime, and we have already spoken of it as a cubic or solid second (§ 13). Of these there are three.

The smallest size is a cube, *AFEH*, measuring 1 inch each way, that is, a cubic inch.

Thus the cube $ABCD$ represents a volume equal to the sum of—

1. One cubic foot.
2. Three cubic primes.
3. Three cubic seconds.
4. One cubic inch or cubic third.

It will be found that the above result may be obtained by the following scheme of work, in which we multiply each term in the length of the solid by each term in the breadth, and then each term in this product by each term in the depth.

1 ft. 1 in.			
1 ft. 1 in.			
1 × 1 sq. ft.	1 × 1 sup. primes		
	1 × 1 sup. primes	1 × 1 sq. in.	
1 sq. ft.	2 sup. primes	1 sq. in.	
1 ft.	1 in.		
1 × 1 cub. ft.	2 × 1 cub. primes	1 × 1 cub. sec.	
	1 × 1 cub. prime	2 × 1 cub. sec.	1 × 1 cub. in.
1 cub. ft.	3 cub. primes	3 cub. secs.	1 cub. in.

In this scheme of work we assume the law that, in any rectangular solid—

Square inches in base × *linear feet in depth* } = *cub. secs. in volume.*

Superficial primes in base × *linear feet in depth* } = *cub. primes in volume.*

Superficial primes in base × *linear inches in depth* } = *cub. secs. in volume.*

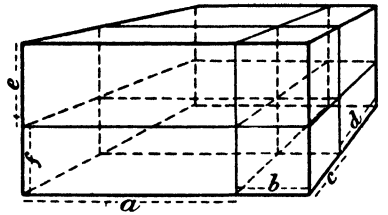
Square feet in base × *linear inches in depth* } = *cub. primes in volume.*

The proof of this law is simple, and is left as an exercise for the student.

The scheme of work in the above example may be shortened thus:

1 ft.	1′		
1 ft.	1′		
1	1′		
	1′	1″	
1	2′	1″	
1	1′		
1	2′	1″	
	1′	2″	1‴
1	3′	3″	1‴

The plan of work here depends upon that property of a rectangular solid, whose length measures $(a + b)$ units, breadth $(c + a)$ units, and depth $(e + f)$ units, viz. that its volume is the sum of eight rectangular solids measuring—



$a \times c \times e$ solid units
$b \times c \times e$ „
$a \times d \times e$ „
$b \times d \times e$ „
$a \times c \times f$ „
$b \times c \times f$ „
$a \times d \times f$ „
$b \times d \times f$ „

If the dimensions of a rectangular solid involve twelfths of an inch, and we wish to find its volume by the method of duodecimals, the principle will be seen to be exactly the same.

Remembering, then, that—

1 standard unit = 12 primes
1 prime = 12 seconds
1 second = 12 thirds
1 third = 12 fourths

and so on, whether the standard unit be linear, superficial, or cubic, and remembering that *the order of a product is the sum of all the orders of its factors* (see Chapter III.), we shall find no difficulty in understanding the following Illustrative Examples.

ILLUSTRATIVE EXAMPLES.

123. *Example* 1.—Find by duodecimals the volume of a rectangular solid which measures 5 ft. 8 in. by 3 ft. 6 in. by 4 ft. 10 in.

Volume of rectangular solid = 5 ft. 8 in. × 3 ft. 6 in. × 4 ft. 10 in. . § 115.

ft.	in.		
5	8		
3	6		
17	0′		
2	10′	0″	
19	10′	0″	
4	10′		
79	4′	0″	
16	6′	4″	0‴
95	10′	4″	0‴

∴ Volume of rectangular solid = 95 cub. ft. 10 cub. primes 4 cub. seconds

Example 2.—Find by duodecimals the volume of a rectangular solid which measures 3 ft. 4 in. 5 twelfths of an inch by 2 ft. 7 in. 9 twelfths of an inch by 2 ft. 8 in. 4 twelfths of an inch.

Volume of rectangular solid } = 3 ft. 4 in. 5 twelfths × 2 ft. 7 in. 9 twelfths × 2 ft. 8 in. 4 twelfths . . . . § 115.

3 ft.	4′	5″				
2 ft.	7′	9″				
6	8′	10″				
1	11′	6″	11‴			
	2′	6″	3‴	9ⁱᵛ		
8	10′	11″	2‴	9ⁱᵛ		
2	8′	4″				
17	9′	10″	5‴	6ⁱᵛ		
5	11′	3″	5‴	10ⁱᵛ	0ᵛ	
	2′	11″	7‴	8ⁱᵛ	11ᵛ	0ᵛⁱ
24	0′	1″	7‴	0ⁱᵛ	11ᵛ	0ᵛⁱ

∴ Volume of rectangular solid } = 24 cub. ft. 1 cub. second 7 cub. thirds 11 cub. fifths

Examples—XXI.

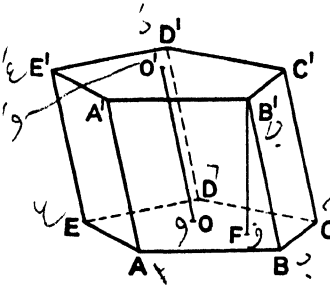
Find, by duodecimals, the volumes of the rectangular solids having the following dimensions :—

1. 3 ft. 2 in. ; 2 ft. 6 in. ; 2 ft.
2. 4 ft. 9 in. ; 3 ft. 5 in. ; 2 ft. 7 in.
3. 3 ft. 6 in. 4 twelfths ; 3 ft. 2 in. 9 twelfths ; 2 ft. 7 in.
4. 4 ft. 7 in. 8 twelfths ; 3 ft. 9 in. 10 twelfths ; 2 ft. 4 in. 6 twelfths.
5. 3 ft. 2 in. 10 twelfths ; 2 ft. 9 in. 7 twelfths ; 1 ft. 7 in. 6 twelfths.
6. 9 in. ; 4 ft. ; 7 twelfths.

CHAPTER XXII.

ON PRISMS, CYLINDERS, AND RINGS.

124. A *prism* is a solid whose sides are parallelograms, and whose ends lie in parallel planes.

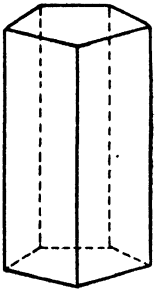


The end on which a prism may be supposed to stand is called the *base* of the prism.

The perpendicular distance between the ends of a prism is called the *height* of the prism.

The straight line joining the middle points of the ends of a prism is called the axis of the prism.

The *length* of a prism is that portion of the axis that lies between the parallel ends.



Thus, in the prism $ABD'E'$—

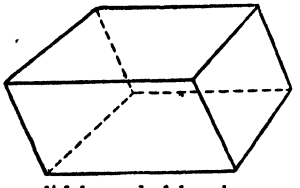
$ABCDE$ is the base,

$B'F$ is the height,

OO' is the length.

A prism is called a *regular* prism when its ends are regular figures.

A prism is called a *right* prism when its edges formed by side faces adjacent to one another are perpendicular to its ends (see figure). Otherwise it is said to be *oblique*.

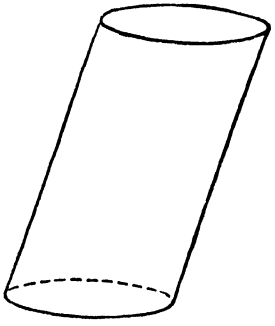


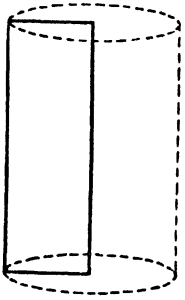
It follows from this that the sides of a right prism are rectangles. All rectangular solids are right prisms.

When the ends of a prism are parallelograms, the prism is called a *parallelopiped* (see figure). Hence a *parallelopiped* may be defined as a solid bounded by three pairs of parallel plane faces.

125. When the number of the sides of a prism is indefinitely increased, and the breadth of each side indefinitely diminished (the perimeter of the cross-section always remaining finite), the surface of the prism tends to become the surface of a *cylinder.*

Hence a *cylinder* may be defined as the limit of a prism, the





number of whose sides is indefinitely increased while the breadth of each side is indefinitely diminished.

The base of a *circular cylinder* is a circle.

A *right circular cylinder* may be seen to be generated by the revolution of a rectangle round one of its sides (see figure).

126. The definition of a prism may be extended so as to include the limiting case of a cylinder, thus—

A *prism* is a solid whose ends are any two parallel plane figures, equal, similar, and similarly placed, and whose sides are determined by straight lines connecting the corresponding points in the circumferences of the two ends (Elliot). Fluted columns and the masonry of arched bridges are familiar examples of prisms in the wider sense of the term.

PROPOSITION XXX.

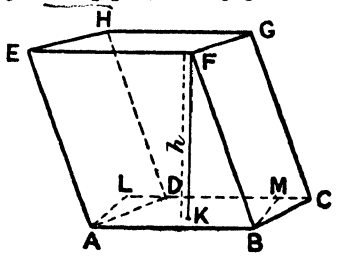
127. *To find the volume of a parallelopiped, having given the area of its base and its height.*

Let *ABGH* be a parallelopiped.

Let its base *ABCD* measure *A* of any square unit.

Let its height *FK* measure *h* of the corresponding linear unit.

It is required to find the volume of *ABGH* in terms of *A* and *h*.



In the plane $ABCD$, and between the parallels AB and DC, draw the rectangle $ABML$.

Then the area of the rectangle $ABML$ is equal to the area of the parallelogram $ABCD$ . . . . . Euc. I. 35.

Now, since parallelopipeds which have equal bases and equal heights are equal in volume . . . . . Euc. XI. 31.

Therefore the volume of the parallelopiped $ABGH$ is equal to the volume of the rectangular solid standing on the base $ABML$ and of height FK.

But the volume of this rectangular solid $= A \times h$ solid units . § 115.

$\therefore$ the volume of the parallelopiped $ABGH = A \times h$ solid units

Hence rule—

Multiply the number of any square unit in the base of a parallelopiped by the number of the corresponding linear unit in the height, then the product will give the number of the corresponding solid unit in the volume.

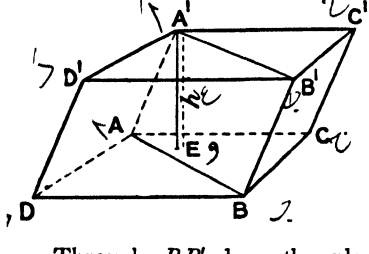
Or briefly—

Volume of parallelopiped = base $\times$ height

$$V = A \times h$$

PROPOSITION XXXI.

128. *To find the volume of a triangular prism, having given its base and height.*



Let ABC' be a triangular prism.

Let its base ABC measure A of any square unit.

Let its height $A'E$ measure h of the corresponding linear unit.

It is required to find the volume of ABC' in terms of A and h.

Through BB' draw the plane $BB'D'$ parallel to the plane $CC'A'$.

Through AA' draw the plane $AA'D'$ parallel to the plane $CC'B'$.

Then the solid $DBC'A'$ will be a parallelopiped. And its volume is double the volume of the prism ABC' , Euc. XI. 28.

But vol. of $DBC'A'$ = area of base $DBCA \times A'E$. § 127.

$\therefore$ vol. of $ABC' = \frac{1}{2}$ area of base $DBCA \times A'E$
$=$ area of triangle $ABC \times A'E$. Euc. I. 34.
$= A \times h$ solid units

Hence rule—

Multiply the number of any square unit in the base of a triangular prism by the number of the corresponding linear unit in the height, then the product will give the number of the corresponding solid unit in the volume.

Or briefly—

Volume of triangular prism = base × height
$$V = A \times h$$

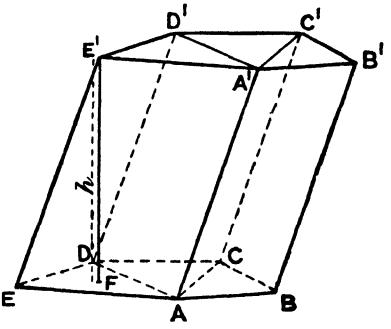
PROPOSITION XXXII.

129. *To find the volume of* any *prism, having given its base and height.*

Let $ABD'E'$ be a prism.

Let its base $ABCDE$ measure A of any square unit.

Let its height $E'F$ measure h of the corresponding linear unit.



It is required to find the volume of $ABD'E'$ in terms of A and h.

Divide the solid into *triangular* prisms by planes through AA'.

These prisms will all have the same height, h, and their bases will be the triangles ABC, ACD, ADE. Hence, if A_1, A_2, A_3, be the areas of these three triangles respectively, and if V_1, V_2, V_3, be the volumes of the three triangular prisms ACB', ADC', AED' respectively, we have—

$$\left.\begin{aligned} V_1 &= A_1 h \\ V_2 &= A_2 h \\ V_3 &= A_3 h \end{aligned}\right\} \quad \ldots\ldots\ldots \S 128.$$

$$\therefore V_1 + V_2 + V_3 = (A_1 + A_2 + A_3)h$$
$$\text{or } V = Ah$$

Hence rule—

Multiply the number of any square unit in the base of a prism by the number of the corresponding linear unit in the height, then the product will give the number of the corresponding solid unit in the volume.

Or briefly—

$$\text{Volume of any prism} = \text{base} \times \text{height}$$
$$V = A \times h \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

Hence—

$$\text{Base of any prism} = \frac{\text{volume}}{\text{height}}$$
$$A = \frac{V}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

And—

$$\text{Height of any prism} = \frac{\text{volume}}{\text{base}}$$
$$h = \frac{V}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii.)}$$

PROPOSITION XXXIII.

130. *To find the volume of a prism, having given its cross-section and length.*

Let $ABC'D'$ be a prism (Fig. 1).

Let its cross-section $A''B''C''D''$ measure A_1 of any square unit.

Let its length AA' measure l of the corresponding linear unit.

It is required to find the volume of $ABCD'$ in terms of A_1 and l.

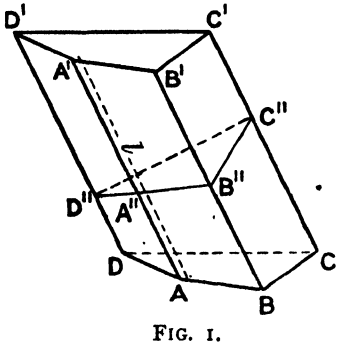


FIG. 1.

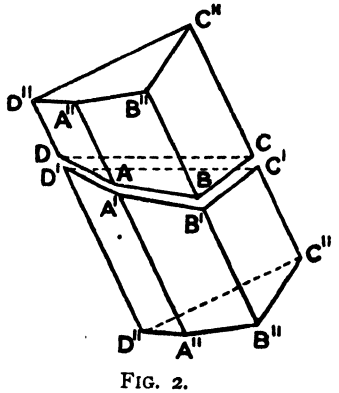


FIG. 2.

Apply the lower portion $ABC''D''$ of the prism $ABC'D'$ to the upper portion $A''B''C'D'$, so that the face $ABCD$ may coincide with the face $A'B'C'D'$ (Fig. 2).

The solid thus formed will be a right prism having $A''B''C''D''$ for its base and $A''A''$ ($= AA'$ in Fig. 1) for its length or height.

And the volume of this right prism will contain $A_1 l$ solid units . . . . . § 129.

But the volume of this right prism is evidently the same as the volume of the original prism $ABC'D'$.

$\therefore$ the vol. of the original prism $ABC'D' = A_1 l$ solid units

Hence rule—

Multiply the number of any square unit in the cross-section of a prism by the number of the corresponding linear unit in the length, then the product will give the number of the corresponding solid unit in the volume.

Or briefly—

$$\text{Volume of a prism} = \text{cross-section} \times \text{length}$$

$$V = A_1 l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i)$$

Hence—

$$\text{Cross-section of prism} = \frac{\text{volume}}{\text{length}}$$

$$A_1 = \frac{V}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (ii.)$$

And—

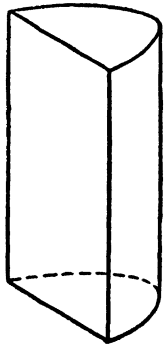
$$\text{length of prism} = \frac{\text{volume}}{\text{cross-section}}$$

$$l = \frac{V}{A_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (iii.)$$

PARTICULAR CASE.

131. Cylinder.

Here the number of sides of the prism is indefinitely increased. But whatever be the number of sides of the prism—



Volume of prism = base × height . § 129.

$\therefore$ volume of cylinder = base × height

$$V = A \times h$$

For a *circular* cylinder, this formula may be written—

$$V = \pi r^2 \times h$$

where r linear units = radius of base.

132. Consider a segment of a prism or of a cylinder made by a plane parallel to the axis (see figure). It follows, from what has been said about the prism and the cylinder, that the volume of such a segment will be determined by the formula—

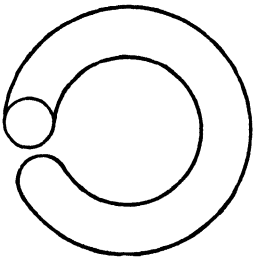
$$V = Ah$$

where A square units = area of end of segment,
and h linear units = height of segment.

Rings.

133. A cylindrical ring may be roughly described as a right circular cylinder bent round in a circle until its ends meet.

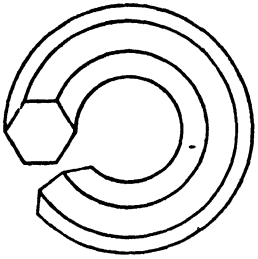
Since in bending the cylinder to form a ring the inner portion is as much contracted as the outer portion is expanded, the volume of the ring may be seen to be the same as the volume of the original cylinder. Hence the volume of a cylindrical ring is equal to the volume of a right circular cylinder whose base is the same as the cross-section of the ring, and whose height is equal to the length of the ring.

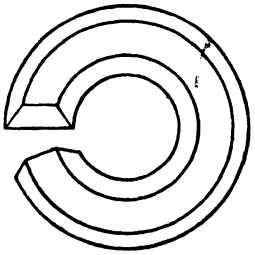


That is—

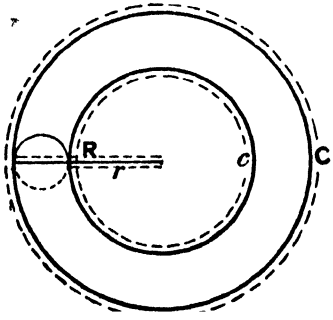
Vol. of a cylindrical ring = area of cross-section × length of ring

$$V = A \times l$$





Note.—By the *length* of a ring is meant its mean circumference or circumference midway between its inner and outer circumferences.



The same reasoning applies to the case of any ring whose cross-section is a figure symmetrical about a line in its own plane perpendicular to the plane of the ring—that is to say, having the outer half corresponding to the inner, as in the figures.

134. In the particular case of the *cylindrical* ring, the following formulæ may be easily verified :—

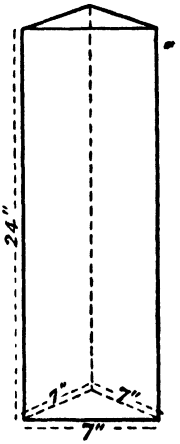
$$V = \frac{\pi^2}{4}(R + r)(R - r)^2$$

$$V = \frac{1}{32\pi}(C + c)(C - c)^2$$

where V is the volume, R and r are the outer and inner radii respectively, C and c the corresponding circumferences.

ILLUSTRATIVE EXAMPLES.

135. *Example* 1.—The base of a right prism is an equilateral triangle with a side of 7 in., and its height is 24 in.: find its volume.



Volume of prism = $A \times h$ cub. in. . § 128.

where $A = \frac{(7)^2\sqrt{3}}{4}$ . . . . . § 21.

$h = 24$;

$\therefore$ volume of prism $= \frac{49 \times 24 \times \sqrt{3}}{4}$ cub. in.

$= 509{\cdot}2$ cub. in.

Example 2.—Find the cost, at the rate of 4*d.* per cubic yard, of digging a pit whose dimensions at the top are 34 ft. 4 in. by 30 ft., and whose depth is 13 ft. 6 in., the sides sloping at an angle of 45°, and the ends being vertical.

Let $ABCD$ represent a cross-section of the pit.

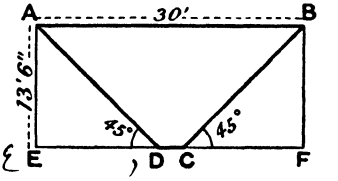
Then $ED = CF = AE = 13\frac{1}{2}$ ft.

$\therefore DC = (30 - 13\frac{1}{2} \times 2)$ ft.

$= 3$ ft.

$\therefore$ area of trapezoid $ABCD$ $= \frac{1}{2}(30+3) \times 13\frac{1}{2}$ sq. ft. . . . § 39.

$\therefore$ cubical content of pit $= \frac{33 \times 27}{2 \times 2} \times 34\frac{1}{3}$ cub. ft. . . § 129.



$\therefore$ cost of digging $= \frac{33 \times 27 \times 103 \times 4}{2 \times 2 \times 3 \times 27}$ pence.

$= 1133$ pence

$= £4\ 14s.\ 5d.$

Example 3.—Find the number of tons of copper in a wire, length 3000 miles, diameter $\frac{1}{8}$ in. Copper wire weighs 555 lbs. per cubic foot.

Volume of wire $= Ah$ cub. ft. . . . . § 131.

where $A = \pi\left(\frac{1}{16 \times 12}\right)^2$,

$h = 3000 \times 1760 \times 3$;

$\therefore$ volume of wire $= \frac{\pi \times 3000 \times 1760 \times 3}{16 \times 16 \times 12 \times 12}$ cub. ft.

$\therefore$ weight of wire $= \frac{22 \times 3000 \times 1760 \times 3 \times 555}{7 \times 16 \times 16 \times 12 \times 12 \times 112 \times 20}$ tons

$= 334\frac{14983}{23088}$ tons

Example 4.—A reservoir is supplied from a pipe 6 in. in diameter. How many pipes of 3 in. diameter would discharge the same quantity, supposing the velocity the same?

Let V cub. in. = volume of water contained in 1 in. of the 6-in. pipe.

Then $V = \pi \times 3^2$ . . . . . § 131.

Let v cub. in. = volume of water contained in 1 in. of the 3-in. pipe.

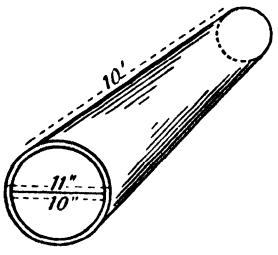
Then $v = \pi \times (\frac{3}{2})^2$ . . . . § 131.

But number of pipes required $= \frac{V}{v}$

$\therefore$ " " $= \frac{\pi \times 3^2}{\pi \times (\frac{3}{2})^2}$

$= 4$

Example 5.—Find the weight of iron in a pipe whose interior and exterior diameters measure 10 in. and 11 in. respectively, and length 10 ft., a cubic inch of iron weighing 0·26 lb. ($\pi = 3{\cdot}1416$)



Volume of iron $= (A_1 - A_2)h$ cub. in. § 131.

where $A_1 = \pi \times (\frac{11}{2})^2$ . . . . § 71.

$A_2 = \pi \times (\frac{10}{2})^2$ . . . . § 71.

$h = 120$;

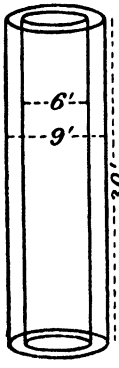
$\therefore$ volume of iron $= \pi\{(\frac{11}{2})^2 - (\frac{10}{2})^2\} \times 120$ cub. in.

$= \frac{\pi \times 21 \times 120}{4}$ cub. in.

$\therefore$ weight of iron $= \frac{3{\cdot}1416 \times 21 \times 120 \times 0{\cdot}26}{4}$ lbs.

$= 514{\cdot}594$ lbs.

Example 6.—A well is to be dug 30 ft. deep, and lined with a circular ring of masonry $1\frac{1}{2}$ ft. in thickness; the interior diameter of well when completed will be 6 ft. Find the total cost of the work, the excavation costing at the rate of Rs. 5 per 100 cub. ft., and the brickwork Rs. 25 per 100 cub. ft.



Volume of earth excavated $= A \times h$ cub. ft. § 131.

where $A = \pi \times (4\frac{1}{2})^2$, ... § 71.

and $h = 30$;

$\therefore$ volume of earth excavated $= \frac{\pi \times 81 \times 30}{4}$ cub. ft.

$\therefore$ cost of excavating earth $= \frac{\pi \times 81 \times 30 \times 5}{4 \times 100}$ Rs.

$= \frac{\pi \times 243}{8}$ Rs.

Volume of masonry $= (A_1 - A_2)h$ cub. ft. . . § 131.

where $A_1 = \pi \times (4\frac{1}{2})^2$ . . . . . § 71.

$A_2 = \pi \times 3^2$ . . . . . § 71.

$h = 30$;

$\therefore$ volume of masonry $= \pi\{(4\frac{1}{2})^2 - 3^2\}30$ cub. ft.

$$\therefore \text{cost of masonry} = \frac{\pi \times 3 \times 15 \times 30 \times 25}{2 \times 2 \times 100} \text{ Rs.}$$

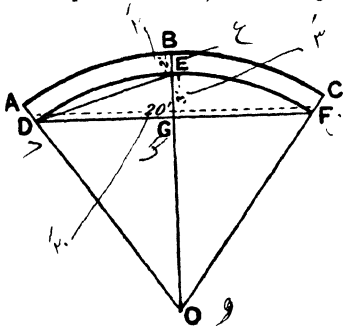
$$= \frac{\pi \times 675}{8} \text{ Rs.}$$

$$\therefore \text{total cost} = \pi\left(\tfrac{242}{8} + \tfrac{675}{8}\right) \text{ Rs.}$$

$$= \tfrac{22}{7} \times \tfrac{917}{8} \text{ Rs.}$$

$$= \text{Rs. } 360 - 10 \text{ as.} - 3\tfrac{3}{7} \text{ p.}$$

Example 7.—A bridge arch has a span of 20 ft., a rise of 3 ft., depth of voussoir 2 ft., and its length from face to face is 30 ft.: find how many cubic feet of masonry it contains.



Take the figure as a cross section of the arch.

Let $OG = x$ ft.

Then $3(2x + 3) = 100$ § 75.

$$x = \tfrac{91}{6}$$

$\therefore$ radius of arc $DEF = \tfrac{109}{6}$ ft.

Again—

$$DE = \sqrt{10^2 + 3^2} \text{ ft. § 16.}$$

$$= \sqrt{109} \text{ ft.}$$

$\therefore$ arc $DEF = \tfrac{1}{3}\{8\sqrt{109} - 20\}$ ft. . . . § 81.

$\therefore$ area of sector $ODEF = \tfrac{1}{2} \times \tfrac{109}{6} \times \tfrac{1}{3}\{8\sqrt{109} - 20\}$ sq. ft. § 86.

$$= \tfrac{1}{2} \times \tfrac{109}{6} \times 21{\cdot}174 \text{ sq. ft.}$$

Now, sectors $ODEF$ and $OABC$ are similar figures;

$\therefore$ area of $OABC$: area of $ODEF = OB^2 : OE^2$ § 104.

that is—

$$\text{area of } OABC : \tfrac{1}{2} \times \tfrac{109}{6} \times 21{\cdot}174 \text{ sq. ft.} = (\tfrac{121}{6})^2 : (\tfrac{109}{6})^2$$

$$\text{area of } OABC = \tfrac{1}{2} \times \tfrac{109}{6} \times 21{\cdot}174 \times \frac{(121)^2}{(109)^2} \text{ sq. ft.}$$

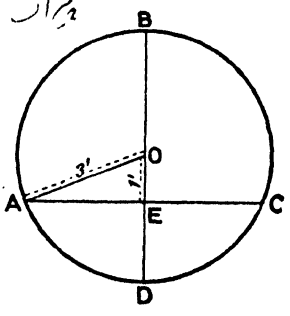
$$\therefore \text{area of section} = \tfrac{1}{2} \times \tfrac{109}{6} \times 21{\cdot}174 \times \left\{\frac{(121)^2}{(109)^2} - 1\right\} \text{ sq. ft.}$$

$$= 44{\cdot}68 \text{ sq. ft.}$$

$$\text{and volume of masonry} = 44{\cdot}68 \times 30 \text{ cub. ft.} \quad . \quad . \quad . \quad . \quad . \quad \text{§ 129.}$$

$$= 1340{\cdot}4 \text{ cub. ft.}$$

Example 8.—A cylindrical buoy, 8 ft. long and 6 ft. in diameter, floats with its axis horizontal and 1 ft. above the surface of the water: find the weight of the buoy.



Let $ABCD$ represent a cross-section of the floating buoy, and AC the level of the water.

Then $OA = 3$ ft.

$OE = 1$ ft.

and $AE = \sqrt{9 - 1}$ ft

$= \sqrt{8}$ ft. . § 16.

Also segment $AECD = \frac{4}{3} h\sqrt{(\frac{1}{4}c^2 + \frac{2}{5}h^2)}$ sq. ft. . § 90.

where $h = 2$,

$c = 2\sqrt{8}$;

$\therefore$ segment $AECD = \frac{4}{5} \times 2 \times \sqrt{(\frac{1}{4} \times 32 + \frac{2}{5} \times 4)}$ sq. ft.

$= \frac{32\sqrt{15}}{15}$ sq. ft.

Now, volume of solid immersed = Ah cub. ft. . . § 132.

where $A = \frac{32\sqrt{15}}{15}$,

$h = 8$;

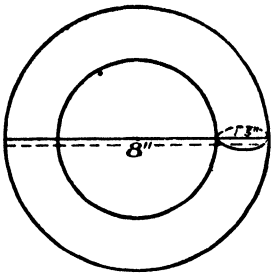
$\therefore$ volume of solid immersed = $\frac{256\sqrt{15}}{15}$ cub. ft.

$\therefore$ water displaced = $\frac{256\sqrt{15}}{15}$ cub. ft.

But weight of buoy = weight of water displaced and 1 cub. foot of water weighs $62\frac{1}{2}$ lbs. . . . § 111.

$\therefore$ weight of buoy = $\frac{256\sqrt{15} \times 125}{15 \times 2}$ lbs.

= 4131 lbs. nearly



Example 9.—The thickness of a solid cylindrical ring is 1·5 in., and its outer diameter 8 in.: find its volume. ($\pi = 3{\cdot}1416$.)

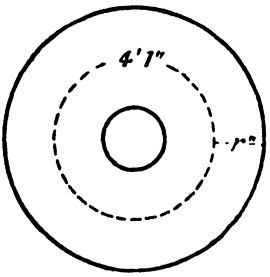
Volume of ring = Al cub. in. § 133.

where $A = \pi\left(\frac{1{\cdot}5}{2}\right)^2$ . . . . . § 71.

$l = \pi(8 - 1{\cdot}5)$ . . . . . § 69.

$\therefore$ volume of ring = $\pi(0{\cdot}75)^2 . \pi(6{\cdot}5)$ cub. in.

= 36·085 cub. in.



Example 10.—The volume of a cylindrical ring is 1386 cub. in., and the length is 4 ft. 1 in.: find the diameter of the cross-section.

Let r in. = radius of cross-section.

Then πr^2 sq. in. = area of cross-section . . § 71.

$\therefore \pi r^2 \times 49 = 1386$ . . § 133.

$r^2 = \frac{1386 \times 7}{22 \times 49} = 9$

$\therefore r = 3$

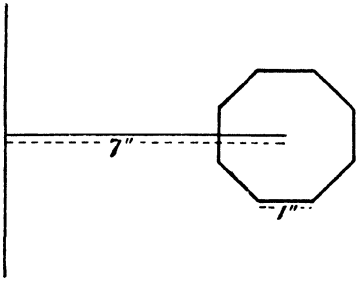
Hence diameter of cross-section = 6 in.

Example 11.—A ring is formed by the revolution of a regular octagon of side 1 in. about an axis in its plane. If the central point of the octagon is 7 in. distant from the axis, find the volume of the ring.

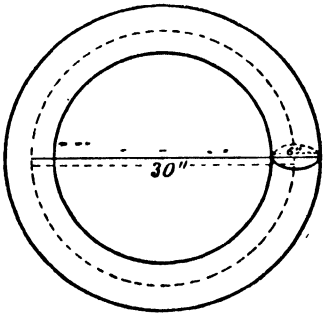
Volume of ring = Al cub. in. . . . . § 133.

where $A = 2 \times 1^2 \times (1 + \sqrt{2})$ . . . . . § 45.
$l = 2 \cdot \pi \cdot 7$ . . . . . § 69.

$\therefore$ volume of ring $= 2(1 + \sqrt{2}) \cdot 2 \cdot \frac{2}{7}^2 \cdot 7$ cub. in.
$= 212{\cdot}45$ cub. in.



Example 12.—The cross-section of a solid ring is an ellipse whose major and minor axes are 6 in. and 4 in. respectively. The mean diameter of the ring is 30 in. It is made of material weighing 30 lbs. per cub. foot. Find its weight. ($\pi = 3{\cdot}1416$.)

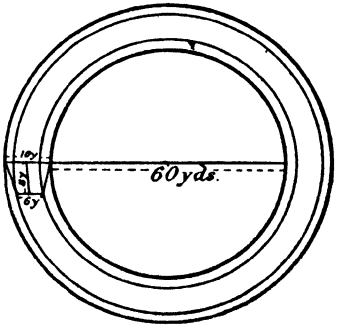


Volume of ring $= Al$ cub. in. § 133
where $A = \pi \cdot \frac{6}{2} \cdot \frac{4}{2}$,
$l = \pi \cdot 30$;

$\therefore$ volume of ring $= \pi \times 6 \times \pi \times 30$ cub. in.

$\therefore$ weight $= \dfrac{\pi \times 6 \times \pi \times 30 \times 30}{1728}$ lbs.
$= 30{\cdot}84$ lbs.

Example 13.—A circular entrenchment of 60 yds. diameter is to be surrounded by a ditch 10 yds. wide at top and 6 at bottom, and 8 yds. deep, sloping equally on both sides. How many cubic yards must be excavated? ($\pi = 3{\cdot}1416$.)



The ditch may be regarded as a ring whose cross-section is the cross-section of the ditch, and whose inner diameter measures 60 yds.

Hence volume of ditch $= Al$ cub. yds. § 133
where $A = \dfrac{10 + 6}{2} \times 8$ . . § 39.
$l = \pi \times 70$. . . . . § 69.

$\therefore$ volume of ditch $= \pi \times 70 \times 64$ cub. yds.
$= 14{,}074{\cdot}368$ cub. yds.

Examples—XXII.

Take $\pi = \frac{22}{7}$ unless otherwise stated.

Find the volumes of the following prisms, having—

1. Base 4 sq. ft. 46 sq. in.; height 1 ft. 9 in.
2. Base 9 sq. ft. 30 sq. in.; height 3 ft. 6 in.
3. Base 19 sq. ft. 120 sq. in.; height 2 yds. 1 ft. 3 in.
4. Base 2 sq. yds. 7 sq. ft. 96 sq. in.; height 1 yd. 2 ft. 8 in.

Find the heights of the following prisms, having—

5. Volume 4 cub. ft. 1512 cub. in.; base 3 sq. ft. 36 sq. in.
6. Volume 21 cub. ft. 540 cub. in.; base 7 sq. ft. 108 sq. in.

Find the areas of the bases of the following prisms, having—

7. Volume 23 cub. ft. 1080 cub. in.; height 2 ft. 3 in.
8. Volume 5 cub. yds. 19 cub. ft. 648 cub. in.; height 2 yds. 1 ft. 6 in.
9. A prism stands upon a triangular base whose sides are 13 in., 20 in., 21 in. If its height is 9 in., find its volume.
10. The base of a prism is a quadrilateral ABCD. If its height is 15 in., and if AB = 9 in., BC = 8 in., CD = 3 in., DA = 4 in., and if the angles at A and C are right angles, find the volume of the prism.
11. The base of a prism is a trapezoid whose parallel sides measure 13 ft. and 17 ft. respectively, the distance between them being 9 ft.: find the volume of the prism if its height is 12 ft.
12. A prism stands upon a triangular base. The volume of the prism is 20 cub. ft. 90 cub. in., and the sides of the base are 5 ft. 8 in., 6 ft. 3 in., and 6 ft. 5 in. Find the height of the prism.
13. The cross-section of a prism is a triangle whose sides are 8 ft. 9 in., 9 ft. 8 in., and 11 ft. 11 in. The volume of the prism is 12 cub. yds. 23 cub. ft. 984 cub. in. Find the length of an edge.
14. What weight of water will fill a vessel in the form of a prism whose base is a regular hexagon of side 2 ft., and whose height is 6 ft.?

Find the volumes of the following circular cylinders:—

15. Radius of base 2 ft.; height 4 ft. 3 in.
16. Radius of base 1 ft. 6 in.; height 3 ft. 9 in.
17. Radius of base 1 yd. 2 ft. 3 in.; height 2 yds. 1 ft. 9 in.

Find correct to the tenth part of an inch the radii of the bases of the following circular cylinders, taking $\pi = 3{\cdot}14159$:—

18. Volume 5000 cub. in.; height 2 ft. 6 in.
19. Volume 40 cub. ft.; height 7 ft. 6 in.
20. Volume 1 cub. yd.; height 4 ft. 6 in.
21. Find to the nearest gallon the quantity of water that will fill a ditch having the following dimensions: length 45 ft., breadth at the top 12 ft., breadth at the bottom 9 ft., and depth 6 ft.
22. What is the cubical content of a well 3 ft. 6 in. in diameter and 42 ft. deep? ($\pi = 3{\cdot}1416$.)
23. How many coins $\frac{3}{4}$ of an inch in diameter and $\frac{1}{8}$ of an inch thick must be melted down to form a rectangular solid whose dimensions are 5 in., 4 in., and 3 in.?
24. Find the cost of digging a well 3 ft. in diameter and 26 ft. in depth, at the rate of Rs.7 per cubic yard.
25. Find the length of wire 0·05 of an inch thick that can be drawn out of a cubic foot of gold. ($\pi = 3{\cdot}1416$.)
26. The thickness of a cylindrical shell is 2 in., the diameter of the outer surface 2 ft. 6 in., and the length 10 ft.: find the number of cubic feet of material in the shell. ($\pi = 3{\cdot}1416$.)
27. Find the cost of a pipe whose bore is 3 in., length 6 yds., thickness $\frac{1}{4}$ of an inch, at the rate of 2 annas per cubic inch.

28. Find the edge of a cube whose volume is the same as that of a prism 4 ft. high standing on a triangular base whose sides are 5 ft. 5 in., 5 ft. 5 in., and 9 ft. 4 in.

29. Find the solidity of a prism 5 ft. 4 in. high, and whose base is an equilateral triangle of side 1 ft. 3 in.

30. A solid cylindrical ring is 1 in. thick, and contains 30 cub. in. : find its inner and outer diameters. ($\pi = 3{\cdot}14159$.)

31. If the length and breadth of a cistern are respectively twice and one half the depth, and if the cistern is capable of holding 8000 gallons, find the depth.

32. How many gallons of water flow through a pipe in 20 minutes if the bore of the pipe is 2 in., and if the water flows at the rate of 4 miles an hour?

33. How many cubic yards of earth must be removed to make a railway cutting 2 miles long, 64 ft. wide at the top, 34 ft. at the bottom, and 28 ft. deep?

34 If the specific gravity of marble is 2·7, find the weight of a cylindrical shaft of marble 40 ft. high and 2 ft. in diameter. ($\pi = 3{\cdot}1416$.)

35. Find the thickness of a cylindrical ring whose mean circumference is 2 ft. and solidity 10 cub. in. ($\pi = 3{\cdot}14159$.)

36. A cylindrical vessel holds 500 gallons of water, and its diameter is 5 ft. : find its depth.

37. Find the volume of a right circular cylinder whose height is 6 ft. 6 in. and circumference 5 ft. 4 in.

38. A circular shaft is 75 ft. deep, and 3 ft. 4 in. in diameter : find the cost of sinking it at the rate of 8 annas per cubic foot.

39. The cross-section of an oblique triangular prism is an isosceles triangle of sides 13 ft., 13 ft., and 10 ft. The length of the prism is 2 yds. 2 ft. 9 in. Find its volume.

40. What is the volume of a flat ring whose height is $1\frac{1}{4}$ in., inner circumference 2 ft. 6 in., and outer diameter 10 in.?

41. A right circular cylinder whose length is 1 ft., and the radius of whose base is 6 in., is cut into two segments by a plane parallel to the axis and distant $3\sqrt{3}$ in. from it: find the volume of the smaller segment. ($\pi = 3{\cdot}1416$.)

42. If the plane in Example 41 is distant $3\sqrt{2}$ in. from the axis, find the volume of the smaller segment. ($\pi = 3{\cdot}1416$.)

43. The length of a cylindrical ring is 45 in., and the diameter of the cross-section is $2\frac{1}{2}$ in. : find the volume.

44. The radius of the inner circumference of a cylindrical ring is 9 in., and the diameter of the cross-section is $3\frac{1}{2}$ in. : find the volume.

45. The diameters of the outer and inner circumferences of a cylindrical ring are $10\frac{3}{4}$ in. and $9\frac{1}{2}$ in. respectively: find the volume.

46. The radius of the outer circumference of a cylindrical ring is $3\frac{1}{2}$ in., and the diameter of the cross-section is $\frac{3}{4}$ in. : find the volume.

47. The volume of a cylindrical ring is 1782 cub. in., and the length is 5 ft. 3 in. : find the diameter of the cross-section.

48. The volume of a cylindrical ring is 1 cub. ft. 274 cub. in., and the radius of the cross-section is $3\frac{1}{2}$ in. : find the length.

Examination Questions—XXII.

Take $\pi = \frac{22}{7}$ unless otherwise stated.

Prisms.

A. *Bombay University. Diploma in Agriculture: Second Exam.*

1. A vessel in the shape of a prism on a regular hexagonal base, whose side

is 4 in., is filled with liquid : find to three decimal places of an inch how much the liquid will sink if half a pint is taken away.

2. A subway is to be constructed beneath a railway station from one platform to another, and the horizontal portion of the tunnel, 20 yards long, is to have its cross-section a rectangle surmounted by a semicircle, and its sides and top are to be lined with brick. The total height and breadth, exclusive of the bricks, are 3 yds. and $1\frac{1}{3}$ yds. respectively, and the thickness of the bricks is $4\frac{1}{2}$ in. Find the weight in tons of the bricks required for the work, if each brick contains $\frac{9}{16}$ of a cubic foot and weighs 5 lbs.

3. What are the cubic contents of a shaft the mean section of which is a regular hexagon, $2\frac{1}{2}$ ft. in side, and the height 60 ft. ?

B. *Punjab University : First Exam. in Civil Engineering.*

4. Give the rules for finding the contents of a prism.

C. *Madras University, B.E. Exam.*

5. Find the contents of an elliptical arch 40 ft. span and 8 ft. rise ; thickness of arch $3\frac{1}{2}$ ft. at haunch and $2\frac{1}{2}$ ft. at crown ; width of arch 21 ft.

6. The intrados and extrados of an arch 33 ft. long, 30 ft. span, and 7 ft. 6 in. rise are true semi-ellipses ; the thickness of arch at springing is 3 ft., and at the crown 2 ft. : find the volume of the arch.

D. *Sibpur Apprentice Dept. : Monthly Exam.*

7. The base of a certain prism is a regular hexagon ; every edge of the prism measures 1 ft. : find the volume of the prism.

E. *Sibpur Apprentice Dept. : Annual Exam.*

8. Find the quantity of masonry in the segmental arch of a masonry bridge, whose radius of the intrados or soffit is 20 ft., thickness of arch is 2 ft., length of arch is 30 ft., angle subtended by the arch at the centre is 84°.

F. *Roorkee Engineer : Entrance.*

9. The section of a canal is 32 ft. wide at the top, 14 ft. wide at the bottom, and 8 ft. deep. How many cubic yards were excavated in a mile of the canal ? also, if the surface of the water be 26 ft. wide, what is its depth ?

10. A room 30 ft. wide and 40 ft. long is roofed by an arch having a rise of 4 ft. at the centre. The arch is 2 ft. thick. Find the quantity of masonry in it to the nearest cubic foot.

G. *Roorkee Upper Subordinate : Entrance.*

11. Find the number of cubic feet of arched masonry in a bridge whose dimensions are as follows : span 60 ft., rise 15 ft., depth of masonry 4 ft., and length from face to face 50 ft.

12. A pond whose area is 4 acres is frozen over with ice to the uniform thickness of 6 in. : if a cubic foot of ice weigh 896 ozs. avoirdupois, find the weight of ice on the pond in tons.

13. Find the number of cubic feet of masonry in an arch whose span is 20 ft., rise 3 ft., length from face to face 30 ft., and depth of voussoir $2\frac{1}{2}$ ft.

H. *Roorkee Upper Subordinate : Monthly.*

14. A hollow column is circular inside and elliptical outside ; the axes of the ellipse are $4\frac{1}{2}$ and 5 ft., and the diameter of the circle 4 ft. : find the volume, the column being 30 ft. high.

15. Find the quantity of masonry in a roof arch, and its cost at the rate of

Rs.35 per 100 cub. ft. Dimensions: length of arch 40 ft., span 15 ft., rise 3 ft., and thickness 6 in.

16. What must be the length to the nearest foot of a hospital to accommodate 50 patients? the building to be 24 ft. wide, side walls 20 ft. high, and the rise of the roof, which is gabled, two-sevenths of the span, allowing 1200 cub. ft. of air space to each patient.

I. *Roorkee Engineer: Final.*

17. The span of a bridge is 30 ft., rise to intrados 7 ft. 6 in., thickness of arch 3 ft., length 30 ft.: how many cubic feet of masonry does the arch contain, and what would be the cost of constructing it at the rate of Rs.30 per 100 cub. ft.?

18. Find the quantity of masonry in a bridge arch of 30 ft. span, rise one-fourth of span, thickness of arch 3 ft., and length 21 ft., and the cost of constructing the same at Rs.30 per 100 cub. ft.

19. Find the quantity of earthwork in a section of a bund 100 ft. long, of which the breadth at the top is equal to the height, the inner slope is 3 to 1, and the outer $1\frac{1}{2}$ to 1, and the height of the embankment 15 ft.

20. Find the number of cubic feet of masonry in an arch whose clear span is 20 ft., rise 5 ft., length from face to face 30 ft., and depth of voussoir 18 in.

Cylinders.

A. *Allahabad University: Intermediate.*

21. The trunk of a tree is a right circular cylinder 5 ft. in radius and 30 ft. high: find the volume of the timber which remains when the trunk is trimmed just enough to reduce it to a rectangular parallelopiped on a square base.

B. *Bombay University, L.C.E.: Second Exam.*

22. A cubical foot of brass is drawn into a cylindrical wire $\frac{1}{10}$ of an inch in diameter. This wire is just long enough to go round a circular field: find approximately the area of the field in acres.

C. *Punjab University: First Exam. in Civil Engineering.*

23. A cubic foot of brass is drawn into a wire one-tenth of an inch in diameter: find its length.

24. A well, $7\frac{1}{2}$ ft. inside diameter, is to be sunk 22 ft. deep, with a brick lining of $13\frac{1}{2}$ in. in thickness. Find—

(*a*) Excavation of earthwork.

(*b*) Quantity of brickwork.

25. A well is to be dug 5 ft. diameter clear inside, and 36 ft. in depth (excluding the curb), with a brick lining of 9 in. in thickness. Find—

(*a*) Excavation of earthwork.

(*b*) Quantity of brickwork.

D. *Calcutta University: F.E. Exam.*

26. From a cylindrical tank $4\frac{1}{2}$ ft. in diameter, water is drawn off at the rate of 110 gallons per hour: find (to the tenth of an inch) by how much the surface would be lowered in 27 minutes. ($\pi = 3{\cdot}1416$, and 1 gallon = 277·25 cub. in.)

27. Prove that the volume of material in a hollow cylinder is equal to $\pi h\{r^2 - (r-k)^2\}$, and explain the meaning of the various symbols.

28. A railing is to be constructed of cylindrical posts and two rows of rectangular rails. If the posts be 6 ft. long, 6 in. in diameter, and separated by distances of 7 ft., also if the cross-section of the rails be a rectangle 6 in. by 1 in., find the number of cubic feet of timber required for a railing 1 mile long.

E. *Sibpur Apprentice Dept.: Monthly Exam.*

29. An iron pipe is 3 in. in bore, $\frac{1}{2}$ in. thick, and 20 ft. long: find its weight, supposing that a cubic inch of iron weighs 4·526 ozs.

F. *Sibpur Apprentice Dept.: Annual Exam.*

30. The trunk of a tree is a right circular cylinder, 3 ft. in diameter and 20 ft. high: find the volume of the timber which remains when the trunk is trimmed just enough to reduce it to a rectangular parallelopiped on a square base.

31. Find how many pieces of money $\frac{3}{4}$ in. in diameter and $\frac{1}{8}$ in. thick, must be melted down in order to form a cube whose edge is 3 in. long.

G. *Sibpur Apprentice Dept.: Final Exam.*

32. A well is to be dug 5 ft. inside diameter, and 36 ft. in depth: find the quantity of earth to be excavated, and the quantity of brickwork required for a lining of 10 in. in thickness.

H. *Roorkee Engineer: Entrance.*

33. A hollow circular cylinder of cast iron is 31·43 ft. in circumference, and 9 ft. 9·5 in. in diameter inside: find its thickness. ($\pi = 3{\cdot}14159$.)

34. A cubic foot of brass is to be drawn into a cylindrical wire $\frac{1}{40}$ of an in. in diameter: what will be the length of the wire?

I. *Roorkee Upper Subordinate: Entrance.*

35. Find the weight of a cast-iron pipe whose length is 9 ft., the bore 3 in., and the thickness of the metal 1 in. A cubic inch of cast iron weighs $\frac{1}{4}$ lb.

36. If 1 mile length of copper wire weigh 1 cwt., find the area of a section, copper being 8·96 times as heavy as water, and 1 cub. ft. of water weighing 1000 ozs. avoirdupois.

37. Find how many gallons of water can be held in a leathern hose 2 in. in bore and 40 ft. long.

38. A roller is wanted which must be $3\frac{1}{2}$ ft. in length and weigh 10 maunds. It is to be of freestone of the specific gravity 2·5. What must be its diameter? (1 seer = 2 lbs.)

39. The length of an iron cylindrical vessel with closed ends is 4 ft., its outside circumference is 40 in., and the thickness of the metal 1 in. Find the entire weight when the cylinder is filled with water, iron being $7\frac{1}{2}$ times heavier than water, and water weighing 1000 ozs. per cubic foot.

40. The internal depth and the diameter of a hollow cylinder are respectively 4 ft. $2\frac{1}{2}$ in. and 8 in. A solid cylinder of the same depth and $6\frac{1}{4}$ in. diameter, stands inside it. How many gallons of water can be poured into the remaining space if a gallon contain 277·75 cub. in., and the area of a circle is $\frac{11}{14}$ of the square of its diameter?

41. Find the cubic inches of material in a cylindrical tube, the radius of the outer surface being 10 in., the thickness 2 in., and the height 9 in.

J. *Roorkee Engineer: Final.*

42. The radius of the inner surface of a leaden pipe is $1\frac{1}{2}$ in., and the radius of the outer surface is $1\frac{9}{16}$ in. If the pipe be melted and formed into a solid cylinder of the same length as before, find the radius.

43. Find the quantity of masonry in a well 10 ft. interior diameter, 50 ft. deep; thickness of masonry ring is 18 in.: what would be the cost of constructing the masonry at the rate of Rs.25 per 100 cub. ft.?

44. Find the thickness of lead in a pipe $1\frac{1}{4}$ in. bore which weighs 14 lbs. per yard in length, a cubic foot of lead weighing 11,325 ozs.

45. If a round pillar 7 in. in diameter contain 4 cub. ft. of stone, what is the diameter of a pillar of equal length which contains ten times as much?

46. A well, 10 ft. inner diameter and 40 ft. deep, is to be made; 15 ft. will be through clay, the remainder in rock. The portion in clay will be protected by a masonry ring 18 in. thick. Find what the cost of making it will be at the following rates per 100 cub. ft.: masonry, Rs.20; excavating clay, Rs.3; rock-cutting, Rs.14.

K. *Superior Accounts.*

47. A hollow cylinder of cast iron, 20 ft. in length and 6 ft. in diameter outside, is placed on end and loaded uniformly on the top with a weight of 300 tons: determine the thickness of the metal so that the pressure on the base may be 1 ton per square inch, the weight of a cubic foot of the iron being 441 lbs.

Segments of Cylinders.

A. *Sibpur Apprentice Dept.: Final.*

48. A prism having a square section, each side of which is 10", penetrates into a solid cylinder, the diameter of which is 26"; the axes of the two intersect at right angles: find the volume of that part of the prism which is inside the cylinder.

B. *Roorkee Engineer: Entrance.*

49. A square hole 2 in. wide is cut through a solid cylinder of which the radius is $\sqrt{2}$ in., so that the axis of the hole cuts at right angles the axis of the cylinder: find how much of the material is cut out. ($\pi = 3{\cdot}1416$.)

C. *Roorkee Upper Subordinate: Entrance.*

50. A cylinder of 3 ft. radius, 12 ft. in length, is immersed one third of its depth in water, with its axis horizontal: find the weight of the cylinder.

Rings.

A. *Bombay University, Diploma in Agriculture: Second Exam.*

51. Find the solid contents of a cylindrical ring, whose thickness is 9 and inner diameter 32.

B. *Bombay University, L.C.E.: First Exam.*

52. The inner diameter of a cylindrical ring is 2·5 in., and the outer diameter 3·8 in.: find the volume of the ring and the weight at the rate of 11,000 ozs. to the cubic foot.

C. *Bombay University, L.C.E.: Second Exam.*

53. Investigate the volume of a cylindrical ring.

D. *Calcutta University: F.E. Exam.*

54. A cylindrical ring whose mean diameter is 18 in. weighs 4033⅓ ozs.: find the radius of the transverse circular section if 240 cub. in. of the substance of which it is made weigh 1000 ozs.

E. *Sibpur Apprentice Dept.: Monthly Exam.*

55. The volume of a cylindrical ring is 800 cub. in., the radius of cross-section is 2": find the length of the ring.

F. *Roorkee Engineer: Entrance.*

56. A carriage drive is to be made round the outside of a circular park whose radius is 585 ft.; the metalling is to be 30 ft. wide and 9 in. deep: what will it cost at Rs.6 per 100 cub. ft.?

57. A circular entrenchment 54 yds. in diameter is to be surrounded by a ditch 6 yds. wide at top and 4 yds. wide at bottom, and 5 yds. deep: find the number of cubic feet to be excavated.

G. *Roorkee Upper Subordinate: Entrance.*

58. The volume of a cylindrical ring is 100 cub. in., and the length is 20 in.: find the inner diameter.

Additional Examination Questions—XXII.

59. If there are 277·2738 cub. in. in a gallon of water, how many tanks, each containing 1000 galls. would be completely filled by a rainfall of 1·25 in. upon a field 513·47 ac. in area? (Allahabad University: Intermediate.)

60. Water flows from a tank through a circular pipe at the rate of 30 yds. per minute. If the pipe is 7 in. in diameter, and the tank is rectangular in shape, 40 yds. long by 25 yds. 2 ft. broad, how long will it be before the level of the water falls 3 in.? (Allahabad University: Intermediate.)

61. Find the weight of an iron pipe 10 ft. long, 2 ft. 6 in. in inside diameter, and 1½ in. thick, the specific gravity of iron being 7·14, and the weight of a cubic foot of water 1000 ozs. (Roorkee Engineer: Entrance.)

62. A swimming bath is 20 yds. long and 8 yds. wide, with steps at one end 1 ft. 6 in. wide and 9 in. deep, extending over the whole width of the bath till a depth of 4 ft. 6 in. is reached. Then the bottom slopes down to the other end with an inclination of 1 in 15. Find in gallons the quantity of water the bath can contain when full. (Madras University: B.E. Exam.)

63. A cubic foot of brass is drawn into wire $\frac{1}{16}$ in. in diameter: find the length of the wire. (Madras University: B.E. Exam.)

64. A sheet of metal ⅛ in. thick is made into a pipe whose internal diameter is ½ in. This pipe is placed round a cylinder 1 ft. in radius. Find how many cubic inches of water it will contain, and how many cubic inches of metal are required to make it. (Calcutta University: F.E. Exam.)

CHAPTER XXIII.

ON PYRAMIDS AND CONES.

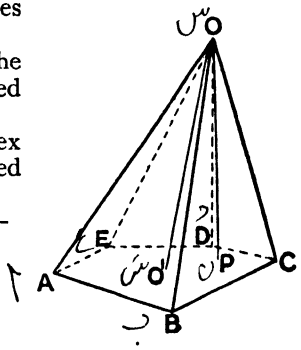
136. A *pyramid* is a solid whose sides are triangles, having a common vertex, and whose base is a plane rectilineal figure.

The common vertex of its side faces is called the *vertex* of the pyramid.

The perpendicular drawn from the vertex of a pyramid to its base is called the *height* of the pyramid.

The straight line joining the vertex to the middle point of the base is called the *axis* of the pyramid.

Thus, in the pyramid *OABCDE*—



ABCDE is the base,
O is the vertex,
OP is the height,
OO' is the axis.

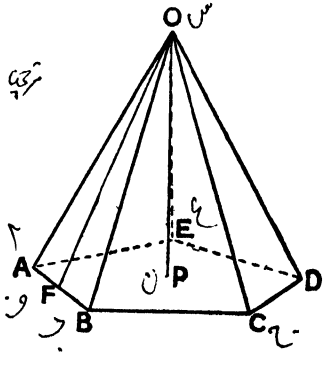
A pyramid is called a *regular* pyramid when its base is any regular figure.

A pyramid is called a *right* pyramid when the foot of the perpendicular from the vertex on to the base is the middle point of the base.

Otherwise it is said to be *oblique*.

The *slant height* of a right regular pyramid is the straight line joining the vertex to the middle point of one of the sides of the base.

Thus, in the right regular pyramid *OABCDE*—



P is the middle point of the base *ABCDE*,
OF is the slant height.

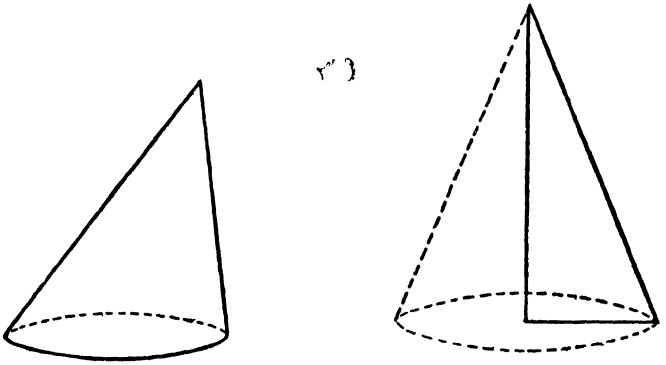
A pyramid is called—

A triangular pyramid or *tetrahedron* when its base is a triangle,

A *square pyramid* when its base is a square,
A *pentagonal pyramid* „ „ pentagon,
A *hexagonal pyramid* „ „ hexagon,
and so on.

137. When the number of the sides of the base of a pyramid is indefinitely increased, and the magnitude of each side indefinitely diminished (the perimeter of the base always remaining finite), the surface of the pyramid tends to become the surface of a *cone.*

Hence a *cone* may be defined as the limit of a pyramid, the



number of the sides of whose base is indefinitely increased, while the magnitude of each side is indefinitely diminished.

The base of a *circular cone* is a circle.

A *right circular cone* may be seen to be generated by the revolution of a right-angled triangle about one of the sides containing the right angle (see figure).

138. The definition of a pyramid may be extended so as to include the limiting case of a cone, thus—

A *pyramid* is a solid whose base is any plane figure, and whose sides are determined by straight lines connecting the various points in the circumference of the base with a common point out of the base, called the vertex (Elliot).

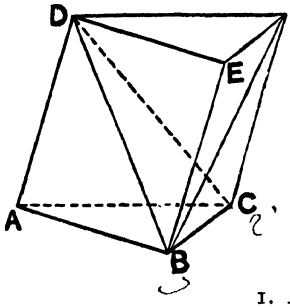
The great pyramid of Egypt, Cheops, is a familiar example of a square pyramid.

PROPOSITION XXXIV.

139. *To find the volume of a tetrahedron, having given its base and height.*

Let $DABC$ be a tetrahedron.

Let its base ABC measure A of any square unit.



Let its height measure h of the corresponding linear unit.

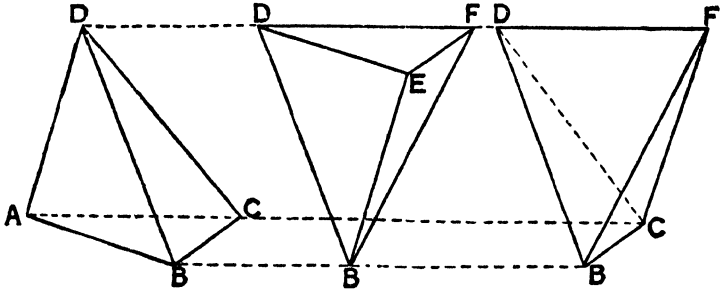
It is required to find the volume of $DABC$ in terms of A and h.

Complete the prism ABF, of which the tetrahedron $DABC$ is a part, as in the figure.

Join BF.

Now, the prism ABF may be seen to be made up of three pyramids—

1. $DABC$.
2. $BDEF$.
3. $BDFC$.



And because $\triangle ABC = \triangle DEF$.

Therefore the pyramids $DABC$ and $BDEF$ may be considered as having equal bases and equal heights.

Also because $\triangle DCF = \triangle ADC$ . . . . . Euc. I. 34.

Therefore the pyramids $BDFC$ and $DABC$ (that is, $BADC$) may be considered as having equal bases and equal heights.

But pyramids on equal bases and of the same heights are equal in volume.[1]

Hence the three pyramids $DABC$, $BDEF$, and $BDFC$ are equal to one another in volume.

$\therefore$ vol. of tetrahedron $DABC = \frac{1}{3} \times$ vol. of prism ABF
$= \frac{1}{3} \times Ah$ solid units . § 128.

Hence rule—

Multiply the number of any square unit in the base of a tetrahedron

[1] The truth of the Proposition "Pyramids on equal bases and of the same heights are equal in volume," follows immediately from Euclid xii. 6, which proves that "Pyramids of the same heights which have polygons for their bases, are to one another as their bases."

by the number of the corresponding linear unit in the height, then one-third the product will give the number of the corresponding solid unit in the volume.

Or briefly—

$$\text{Volume of tetrahedron} = \tfrac{1}{3}\ \text{base} \times \text{height}$$
$$V = \tfrac{1}{3}\,Ah$$

140. Since the foot of the perpendicular from the vertex on to the base of a *regular* tetrahedron is the middle point of the base, and since all the faces of a *regular* tetrahedron are equal and equilateral triangles, it can easily be proved that—

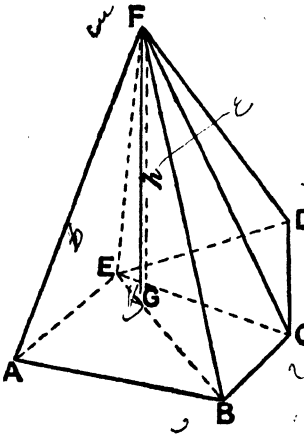
$$\text{(i.) its height} = 2a\sqrt{\tfrac{2}{3}}$$
$$\text{(ii.) its volume} = \frac{2\sqrt{2}}{3} \cdot a^3$$

where $2a$ = the measure of each edge.

These results are useful, and their investigation is left as an exercise for the student.

PROPOSITION XXXV.

141. *To find the volume of* any *pyramid, having given its base and height.*



Let $FABCDE$ be a pyramid.

Let its base $ABCDE$ measure A of any square unit.

Let its height FG measure h of the corresponding linear unit.

It is required to find the volume of $FABCDE$ in terms of A and h.

Divide the solid into *triangular* pyramids by planes through FE.

These pyramids will all have the same height h, and their bases will be the triangles EAB, EBC, ECD.

Hence, if A_1, A_2, A_3 be the areas of these three triangles respectively, and if V_1, V_2, V_3 be the volumes of the three triangular pyramids $FEAB$, $FEBC$, $FECD$ respectively, we have—

$$\left.\begin{aligned} V_1 &= \tfrac{1}{3} \cdot A_1 h \\ V_2 &= \tfrac{1}{3} \cdot A_2 h \\ V_3 &= \tfrac{1}{3} \cdot A_3 h \end{aligned}\right\} \quad \S\,139.$$

$$\therefore V_1 + V_2 + V_3 = \tfrac{1}{3} \cdot (A_1 + A_2 + A_3)h$$

$$\text{or } V = \tfrac{1}{3} \cdot Ah$$

Hence rule—

Multiply the number of any square unit in the base of a pyramid by the number of the corresponding linear unit in the height, then one-third the product will give the number of the corresponding solid unit in the volume.

Or briefly—

$$\textbf{Volume of any pyramid} = \tfrac{1}{3} \times \textbf{base} \times \textbf{height}$$

$$V = \tfrac{1}{3} . Ah \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

Hence—

$$\textbf{Base of any pyramid} = \frac{3 \times \textbf{volume}}{\textbf{height}}$$

$$A = \frac{3V}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

And—

$$\textbf{Height of any pyramid} = \frac{3 \times \textbf{volume}}{\textbf{base}}$$

$$h = \frac{3V}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii.)}$$

PARTICULAR CASE.

142. Cone.

Here the number of sides of the base of the pyramid is indefinitely increased.

But whatever be the number of sides of the base of a pyramid—

Volume of pyramid $= \tfrac{1}{3} \times$ base $\times$ height . . . . § 141.

$\therefore$ volume of cone $= \tfrac{1}{3} \times$ base $\times$ height

$$V = \tfrac{1}{3} . Ah$$

For a *circular* cone this formula may be written—

$$V = \tfrac{1}{3} . \pi r^2 \times h$$

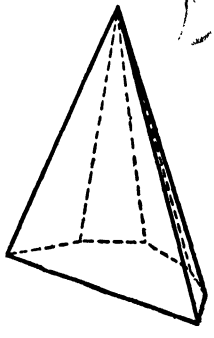
where r linear units = radius of base.

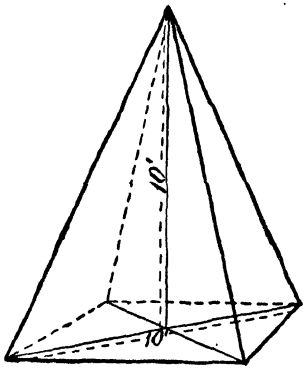
143. Consider a segment of a pyramid or cone made by a plane through the vertex. It follows, from what has been said about the pyramid and cone, that the volume of such a segment will be determined by the formula—

$$V = \tfrac{1}{3} Ah$$

where A square units = area of base of segment,

and h linear units = height of segment.





ILLUSTRATIVE EXAMPLES.

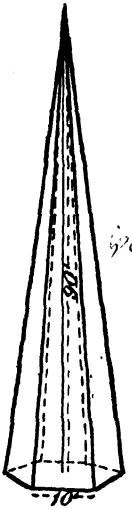
144. *Example* 1.—A right pyramid 10 ft. high has a square base, of which the diagonal is 10 ft.: find its volume.

Volume of pyramid $= \frac{1}{3}Ah$ cub. ft. § 141.

where $A = \frac{1}{2} \times 10 \times 10$. . . § 31.
$h = 10$;

$\therefore$ volume of pyramid $= \frac{1}{3} \times \frac{100}{2} \times 10$ cub. ft.
$= 166\frac{2}{3}$ cub. ft.

Example 2.—Find the volume of a right pyramid on a regular hexagonal base; each side of the base is 10 ft., and the height 90 ft.



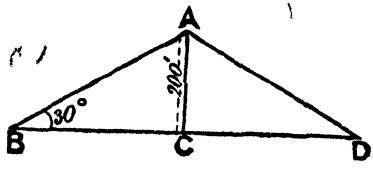
Volume of pyramid $= \frac{1}{3}Ah$ cub. ft. . . . . § 141.

where $A = \frac{3 \times 10^2 \times \sqrt{3}}{2}$ . . . . . § 45.

$h = 90$;

$\therefore$ volume of pyramid $= \frac{1}{3} \times \frac{3 \times 10^2 \times \sqrt{3}}{2} \times 90$ cub. ft.
$= 7794{\cdot}228$ cub. ft.

Example 3.—A right cone is 200 ft. high, and its generating line is inclined at an angle of 30° to the horizon: find its volume.



Let $ABCD$ represent a vertical section of the cone through the axis.

Then BC = radius of base of cone $= \frac{400\sqrt{3}}{2}$ ft. . § 17.

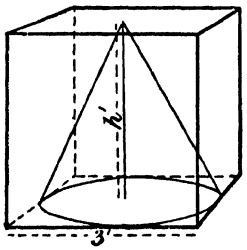
$\therefore$ volume of cone $= \frac{1}{3}Ah$ cub. ft. . . . . § 142.

where $A = \pi\left(\frac{400\sqrt{3}}{2}\right)^2$ . . . . . § 71.

$h = 200$;

$\therefore$ volume of cone $= \frac{1}{3} \times \frac{22}{7} \times \frac{160000 \times 3}{4} \times 200$ cub. ft.
$= 25{,}142{,}857\frac{1}{7}$ cub. ft.

Example 4.—Find the volume of the largest right cone that can be cut out of a cube whose edge is 3 ft. ($\pi = 3{\cdot}1416$.)



The base of the cone will be the circle inscribed in a side of the cube, and the height of the cone will be equal to an edge of the cube.

$\therefore$ volume of cone $= \frac{1}{3}A \times h$ cub. ft. § 142.

where $A = \pi(\frac{3}{2})^2$ . . . . . § 71.

$h = 3$;

$\therefore$ volume of cone $= \frac{1}{3} \times \pi \times \frac{9}{4} \times 3$ cub. ft.

$$= \frac{3{\cdot}1416 \times 9}{4} \text{ cub. ft.}$$

$$= 7{\cdot}0686 \text{ cub. ft.}$$

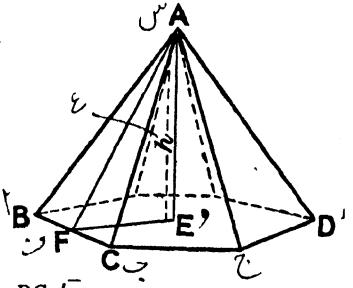
Example 5.—The area of the base of a hexagonal pyramid is $54\sqrt{3}$, and the area of one of its side faces is $9\sqrt{6}$: find the volume of the pyramid.

Let the figure $ABCD$ represent the pyramid.

Then volume of $ABCD = \frac{1}{3}.Ah$ solid units . . § 141.

where $A = 54\sqrt{3}$,

h = number of linear units in AE.



To find AE.

Let BC measure a linear units.

Then $\frac{3a^2\sqrt{3}}{2} = 54\sqrt{3}$. § 45.

$\therefore a = 6$

Again—

$\frac{1}{2}.a \times AF = 9\sqrt{6}$ square units § 20.

$\therefore AF = 3\sqrt{6}$ linear units

Also, $EF = \frac{BC\sqrt{3}}{2}$ . . . . . § 17.

$\therefore EF = 3\sqrt{3}$ linear units

Now, $AE = \sqrt{AF^2 - FE^2}$ . . . . § 16.

$\therefore AE = \sqrt{54 - 27}$ linear units

$= \sqrt{27}$ linear units

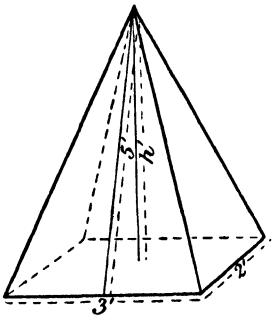
$= 3\sqrt{3}$ linear units

Hence volume of pyramid $= \frac{1}{3} \times 54\sqrt{3} \times 3\sqrt{3}$ solid units

$= 162$ solid units

Example 6.—The base of a pyramid is a rectangle which measures 2 ft. by 3 ft., and the slant height from the vertex to either of the longer sides is 5 ft.: find the height of a cylinder the radius of whose base is 6 in., and of which the solid content is half that of the pyramid.

Vol. of pyramid $= \frac{1}{3}Ah$ cub. ft. . . . . § 141.



where $A = 3 \times 2$ . . . . . § 8.

$h = \sqrt{5^2 - 1^2}$ . . . . . § 16.

$= 2\sqrt{6}$;

$\therefore$ vol. of pyramid $= \frac{1}{3} \times 6 \times 2\sqrt{6}$ cub. ft.

$= 4\sqrt{6}$ cub. ft.

$\therefore$ vol. of required cylinder $= 2\sqrt{6}$ cub. ft.

Hence, if x ft. = height of required cylinder

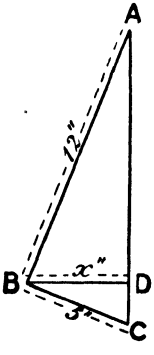
$\pi . (\frac{1}{2})^2 . x = 2\sqrt{6}$ . . . § 131.

$x = \frac{8\sqrt{6}}{\pi}$

$= 6{\cdot}23 \ldots$

height of required cylinder = 6·23 ft. nearly

Example 7.—A right-angled triangle, of which the sides are 5 and 12 in. in length, is made to turn round its hypotenuse : find the volume of the double cone thus formed. ($\pi = 3{\cdot}1416$.)



Let ABC be the right-angled triangle.

From the right angle B draw BD perpendicular to the hypotenuse AC.

Let BD measure x inches.

Then, by similar triangles—

$x : 12 = 5 : 13$ . . . . . § 66.

$x = \frac{60}{13}$

Now—

Vol. of double cone formed by the revolution of ABC round AC $= (\frac{1}{3} . Ah_1 + \frac{1}{3} . Ah_2)$ cub. in. § 142.

where $A = \pi x^2$ . . . . . § 71.

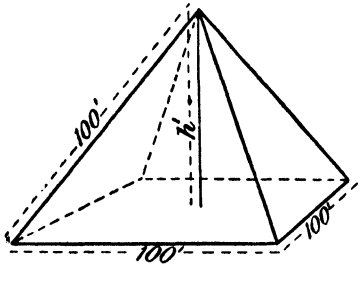
$= \pi(\frac{60}{13})^2$,

$h_1 + h_2 = 13$;

$\therefore$ volume of double cone $= \frac{1}{3} . \pi . (\frac{60}{13})^2 . 13$ cub. ft.

$= 289{\cdot}99$ cub. ft.

Example 8.—A pyramid on a square base has every edge 100 ft. long : find the edge of a cube of equal volume.



Let h feet = height of pyramid.

Then—

Vol. of pyramid $= \frac{1}{3}Ah$ cub. ft. § 141.

where $A = 100 \times 100$ . . § 9.

$h = \sqrt{(100)^2 - (\frac{1}{2} . 100\sqrt{2})^2}$ §§ 16, 17.

$= \sqrt{10{,}000 - 5000}$

$= \sqrt{5000}$

$\therefore$ volume of pyramid $= \frac{1}{3} . 100^2\sqrt{5000}$ cub. ft.

Let a ft. = edge of required cube.

Then $a^3 = \frac{1}{3} . 100^2 . \sqrt{5000}$ . . . . . § 117.

$= 235{,}702{\cdot}25$

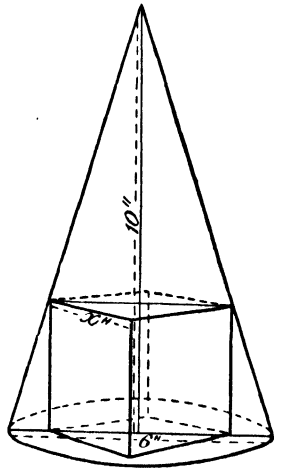
$\therefore a = 61{\cdot}7$ nearly

hence edge of required cube = 61·7 ft. nearly

Example 9.—Find the edge of the greatest cube that can be cut out of a right cone, 10 in. high and 6 in. diameter at the base, the base of the cube to coincide with the base of the cone.

Let x in. = edge of cube.

Then $x\sqrt{2}$ in. = diagonal of face of cube . . § 17.



Therefore by similar triangles—

$10 - x : x\sqrt{2} = 10 : 6$ . . . . . § 66.

$$x = \frac{30}{3 + 5\sqrt{2}}$$

$$= \frac{30(5\sqrt{2} - 3)}{(5\sqrt{2} + 3)(5\sqrt{2} - 3)}$$

$$= \frac{30(5\sqrt{2} - 3)}{50 - 9}$$

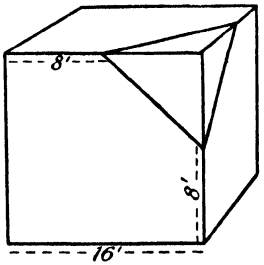
$$= 2{\cdot}978$$

Hence each edge of the greatest cube measures 2·978 in.

Example 10.—Find the volume of the pyramid formed by cutting off a corner of a cube whose edge is 16 ft. by a plane which bisects three conterminous edges.

If we regard one of the three isosceles right-angled triangles whose equal sides each measure 8 ft. as being the base of the pyramid, then the height of this pyramid will measure 8 ft. and its volume

$= \frac{1}{3}Ah$ cub. ft. . . . § 139.



where $A = \frac{1}{2} \times 8^2$

and $h = 8$

hence volume of pyramid

$= \frac{1}{3} \times \frac{1}{2} \times 8^2 \times 8$ cub. ft.

$= 85\frac{1}{3}$ cub. ft.

Examples—XXIII.

(Take $\pi = \frac{22}{7}$, unless otherwise stated.)

Find the volumes of the following pyramids, having—

1. Base 5 sq. ft. 74 sq. in. ; height 2 ft. 6 in.
2. Base 13 sq. ft. 105 sq. in. ; height 3 ft. 3 in.
3. Base 19 sq. ft. 78 sq. in. ; height 6 ft. 2 in.
4. Base 3 sq. yds. 8 sq. ft. 114 sq. in. ; height 2 yds. 2 ft. 10 in.

Find the heights of the following pyramids, having—

5. Volume 8 cub. ft. 1616 cub. in. ; base 2 sq. ft. 98 sq. in.
6. Volume 1 cub. yd. 19 cub. ft. 812 cub. in. ; base 7 sq. ft. 92 sq. in.

Find the areas of the bases of the following pyramids, having—

7. Volume 10 cub. ft. 1080 cub. in. ; height 2 ft. 10 in.
8. Volume 21 cub. ft. 274 cub. in. ; height 2 yds. 2 ft. 5 in.

Find the volumes of the following circular cones, having—

9. Radius of base 7 in. ; height 10 in.
10. Radius of base 2 ft. 11 in. ; height 6 ft.
11. Radius of base 3·5 ft. ; height 4·6 ft.
12. Radius of base 5 ft. 10 in. ; height 1 yd.

Find the radii of the bases of the following circular cones, having—

13. Volume 198 cub. in. ; height 21 in.
14. Volume 132 cub. in. ; height 14 in.
15. Volume 440 cub. in. ; height 7 in.
16. Volume 2200 cub. in ; height 2·3 in.

Find the volumes of the following tetrahedrons, having—

17. Sides of base 51, 37, 20 ft. ; height 10 ft.
18. Sides of base 7, 24, 25 yds. ; height 13 yds.
19. Sides of base 13, 14, 15 ft. ; height 12 ft.
20. Sides of base 35, 44, 75 in. ; height 20 in.

21. Find the volume of a pyramid whose base is an equilateral triangle of side 1 ft., and whose height is 4 ft.

22. The radius of the base of a right circular cone is 6 ft., and the slant height is 6 ft. 6 in. : find the volume.

23. The faces of a pyramid on a square base are equilateral triangles. If each side of the base is 20 in., find the volume of the pyramid.

24. Find to the nearest pound the weight of a cone whose base diameter measures 10 in. and height 15 in., if the material of which it is composed weigh 500 lbs. per cubic foot.

25. The generating line of a right cone is inclined at an angle of 60° to the horizon. If the height of the cone measure 15 in., find its volume.

26. The base of a pyramid is a square of side 54 ft., and its slant height measures 45 ft. : find its volume.

27. Find the volume of the largest right cone that can be cut out of a cube whose edge is 5 in. ($\pi = 3{\cdot}1416$.)

28. Find the edge of the greatest cube that can be cut out of a right cone 1 ft. high and 8 in. diameter at the base, the base of the cube to coincide with the base of the cone.

29. The base of a pyramid is a rectangle which measures 4 yds. 2 ft. 2 in. by 3 yds. 1 ft., and the slant height from the vertex to either of the shorter sides of the base is 4 yds. 1 ft. 1 in. : find the volume.

30. The base of a right pyramid is a regular octagon of side 2 in., and its slant surfaces are inclined to the horizon at an angle of 30° : find the volume.

Examination Questions—XXIII.

(Take $\pi = \frac{22}{7}$.)

Pyramids.

A. *Bombay University, Diploma in Agriculture: Second Exam.*

1. Find the volume of a pyramid when its base is a regular hexagon, each side measuring 6 ft. and height 30 ft.

2. A pyramid is cut out from a cube (edge a) by a plane passing through the extremities of three edges meeting at a corner of the cube: find the volume of the pyramid cut out.

B. *Bombay University: L.C.E. Second Exam.*

3. Two of the side faces of a pyramid are equilateral triangles, and the other two side faces are right-angled triangles: find the volume of the pyramid if the length of each side of the equilateral triangles is 6 ft.

4. The base of a pyramid is a square, and its faces equilateral triangles: prove that its volume is $\frac{a^3}{6}\sqrt{2}$, where a is a side of the base.

C. *Punjab University: First Exam. in Civil Engineering.*

5. A regular hexagonal pyramid has the perimeter of its base 15 ft., and its altitude 15 ft.: find its volume.

6. State the formula for finding the volume of a pyramid.

D. *Calcutta University: F.E. Exam.*

7. Show that the capacity of a tent considered as a prism of n sides surmounted by a pyramid whose heights are respectively h and h_1, and the length of side a is—

$$(3h + h_1)\frac{na^2}{12}\cot\frac{180^\circ}{n}$$

8. The spire of a church is a right pyramid on a regular hexagonal base; each side of the base is 10 ft., and the height is 50 ft. There is a hollow part, which is also a right pyramid on a regular hexagonal base; the height of the hollow part is 45 ft., and each side of the base is 9 ft. Find the number of cubic feet of masonry in the spire.

E. *Sibpur Apprentice Dept.: Monthly Exam.*

9. A pyramid on a square base has four equilateral triangles for its four other faces, each edge being 20 ft.: find the volume.

10. The faces of a pyramid on a square base are equilateral triangles, a side of the base being 120 ft.: find the volume.

11. Find the number of cubic feet in a regular hexagonal room, each side of which is 20 ft. in length, and the walls 30 ft. high, and which is finished above with a roof in the form of a hexagonal pyramid 15 ft. high.

F. *Sibpur Apprentice Dept.: Annual Exam.*

12. Find the volume of the pyramid formed by cutting off a corner of the cube, whose edge is 20 ft., by a plane which bisects its three conterminous edges.

13. A solid is bounded by four equilateral triangles, a side of each triangle being 12 in.: find the volume.

14. The edge of a cube is 14 in.; one of the corners of the cube is cut off, so that the part cut off forms a pyramid with each of its edges terminating in the angle of the cube 6 in. in length: find the volume of the solid that remains.

G. *Sibpur Apprentice Dept.: Final Exam.*

15. Every edge of a pyramid on a triangular base is 1 ft.: show that the volume of the pyramid is $\frac{\sqrt{2}}{12}$ of a cubic foot, and that the volume of any pyramid on a triangular base which has all its edges equal may be obtained by multiplying the cube of an edge by $\frac{\sqrt{2}}{12}$.

H. *Roorkee Engineer: Entrance.*

16. A pyramid whose height is h has for its base a segment of a parabola, the chord of the segment being a, and the perpendicular distance between the chord and the parallel tangent being b: write down the volume of the pyramid, and compare it with that which the pyramid would have if the base were a segment of a circle of the same dimensions. (Area of segment of parabola $= \frac{2}{3} \times$ area of circumscribing parallelogram.)

17. Find the volume of the regular triangular pyramid, a side of its base being 6 ft., and its altitude 60 ft.

18. What is the solidity of a pentagonal pyramid, with a regular base, each side of which is 4 ft., and the altitude of the pyramid 30 ft.?

19. A pyramid on a square base has four equilateral triangles for its four other faces, each edge being 30 feet: find the volume.

I. *Roorkee Engineer: Final.*

20. A pyramid has for its base an equilateral triangle of which each side is 2 ft., and its slant edge is 6 ft.: find its solid content.

J. *Superior Accounts.*

21. The representative gold pyramid in the International Exhibition of 1862, was 10 ft. square at the base, and 44 ft. 9½ in. in height. Find the volume in cubic feet; also the weight if 1 cub. in. of gold weigh 10·14502 ozs. Troy, and the value at 80*s.* per ounce.

Cones.

A. *Punjab University: First Exam. in Civil Engineering.*

22. State the formulæ for finding the volume of a cone.

B. *Sibpur Apprentice Dept.: Monthly Exam.*

23. A right-angled triangle, of which the sides are 3 ft. 6 in. and 5 ft. in length, is made to turn round on the longer side: find the volume of the solid thus formed.

24. The section of a right circular cone by a plane through its vertex perpendicular to the base is an equilateral triangle, each side of which is 12 ft.: find the volume of the cone.

25. A cone 3 ft. high and 2 ft. in diameter at the bottom is placed on the ground, and sand is poured over it until a conical heap is formed 5 ft. high and 30 ft. in circumference at the bottom: find how many cubic feet of sand there are.

26. A pyramid on a regular hexagonal base is trimmed just enough to reduce it to a cone: show that rather less than $\frac{1}{10}$ of the original volume is removed.

C. *Sibpur Apprentice Dept.: Annual Exam.*

27. A right-angled triangle, of which the sides are 3 in. and 4 in. in length, is made to turn round on the longer side: find the volume of the cone thus formed.

D. *Sibpur Apprentice Dept.: Final Exam.*

28. A right-angled triangle, whose remaining angles are 60° and 30°, revolves about its hypotenuse, which is 12 in. long: find the volume of the solid thus described.

29. The sides of a right-angled triangle are 3 in. and 4 in. respectively: find the volume of the double cone formed by the revolution of the triangle round its hypotenuse.

E. *Roorkee Engineer: Entrance.*

30. Find how many gallons are contained in a vessel which is in the form of a right circular cone, the radius of the base being 8 ft. and the slant side 17 ft.

31. Find the solidity of a cone, the diameter of whose base is 3 ft. and its altitude 30 ft.

32. Find the side of the greatest cube that can be cut out of a right cone 10 in. high and 6 in. diameter at the base, the base of the cube to coincide with the base of the cone.

33. The diameter of the base of an oblique cone is 13 ft., the greatest slant height 20 ft., and the least slant height 15 ft.: required the solidity of the cone.

F. *Roorkee Upper Subordinate: Entrance.*

34. A conical tent is required to accommodate 5 people; each person must have 16 sq. ft. of space on the ground, and 100 cub. ft. of air to breathe: give the vertical height, slant height, and width of the tent.

35. A cone whose height is 15 in. and radius of base 6 in. is trimmed sufficiently to reduce it to a pyramid whose base is an equilateral triangle: find the volume of the portion removed.

G. *Roorkee Engineer: Final.*

36. If 6 in. and 1 in. respectively be the radii of two spheres inscribed in a cone so that the greater may touch the less and also the base of the cone, then what will be the volume of that cone?

37. A piece of tin having the form of a quadrant of a circle is rolled up so as to form a conical vessel: required its content, when the radius of the quadrant is 10 in.

38. A right-angled triangle of sides equal to 20 in., 16 in., and 12 in. respectively is made to spin round on its hypotenuse as axis: find the volume of the double cone thus formed.

Additional Examination Question—XXIII.

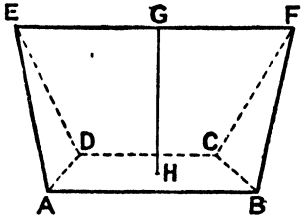
39. How many gallons of water will result from the melting of a pyramid of ice 3 ft. high, and with a hexagonal base of 1 ft. each side, it being given that ice loses 7 % of its volume on melting, and that 1 cub. ft. of ice contains $6\frac{1}{2}$ galls.? (Allahabad University: Intermediate.)

CHAPTER XXIV.

ON WEDGES AND OBLIQUE FRUSTA OF TRIANGULAR PRISMS.

145. A *wedge* is a solid bounded by five plane surfaces, of which the base is a rectangle, the two ends are triangles, and the two sides are trapezoids.

The line in which the two side faces of a wedge intersect is called the *edge* of the wedge.



The edge of a wedge is evidently parallel to the base.

The perpendicular distance between the edge of a wedge and the base is called the *height* of the wedge.

Thus, in the wedge $ABCDEF$—

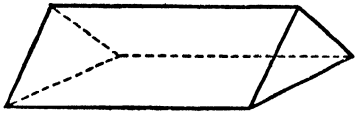
$ABCD$ is the rectangular base,

ADE and BCF are the triangular ends,

$ABFE$ and $CDEF$ are the trapezoidal sides,

EF is the edge,

GH is the height.



When the edge of a wedge is equal in length to that of the base, the wedge may be seen to be a triangular prism (§ 124).

A railway cutting through a ridge may be taken as an example of a wedge.

PROPOSITION XXXVI.

146. *To find the volume of a wedge, having given its edge, its height, and the length and breadth of its base.*

Let $ABCDEF$ be a wedge.

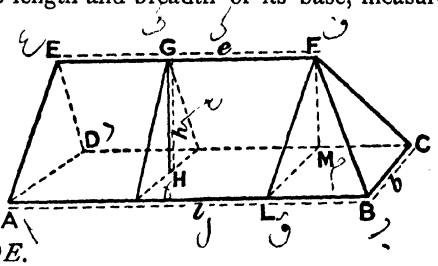
Let its edge EF measure e of any linear unit.

Let its height GH measure h of the same linear unit.

Let AB and BC, the length and breadth of its base, measure l and b of the same linear unit respectively.

It is required to find the volume of $ABCDEF$ in terms of e, h, l, and b.

Divide the wedge into a prism ADF and a pyramid $FLBCM$ by a plane through F parallel to the plane ADE.



Now, volume of prism ADF $= \text{cross section} \times \text{length}$ . . . § 130.

$= \frac{1}{2}bh \times e$ solid units . . . . . § 20

also volume of pyramid $FLBCM$ $= \frac{1}{3}$ base $\times$ height . . . . . § 141.

$= \frac{1}{3}(l - e)b \times h$ . . . . . § 8.

$\therefore$ volume of wedge $= \{\frac{1}{2}bh \times e + \frac{1}{3}(l - e)b \times h\}$ solid units

$$= \left(\frac{3bhe}{6} + \frac{2bhl}{6} - \frac{2bhe}{6}\right) \text{ solid units}$$

$$= \frac{bh}{6}(2l + e) \text{ solid units}$$

Hence rule—

Add the number of any linear unit in the edge of a wedge to twice the number of the same linear unit in the length of the base; multiply the sum by one-sixth of the product of the numbers of the same linear unit in the breadth of the base and in the height of the wedge respectively, then the result will give the number of the corresponding solid unit in the volume.

Or briefly—

$$\textbf{Volume of wedge} = \frac{\textbf{breadth of base} \times \textbf{height of wedge}}{6} \times (2 \times \textbf{length of base} + \textbf{edge of wedge})$$

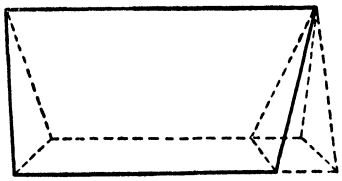
$$\mathbf{V = \frac{bh}{6}(2l + e)}$$

It is important to remember that the length of the base of a wedge is always taken to be that dimension which is parallel to the edge of the wedge.

Note.—Since $\frac{bh}{2}$ square units = area of cross-section, this rule may also be briefly stated thus—

$$\text{Volume of wedge} = \frac{\text{area of cross-section}}{3} \times (2 \times \text{length of base} + \text{edge of wedge})$$

$$V = \frac{A_1}{3}(2l + e)$$



Note.—When the edge of a wedge is longer than the base, the same formula may be proved to hold good by completing the prism, of which the wedge is a part as in the figure, and then regarding the volume of the wedge as the *difference* between the volume of a prism and the volume of a pyramid.

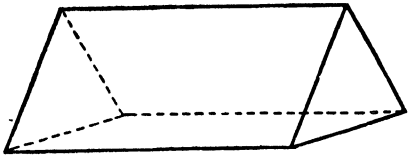
PARTICULAR CASE.

147. Triangular prism.

Here edge of wedge = length of base

$\therefore$ vol. of triangular prism $= \frac{bh}{6}(2l + e)$ solid units . § 146.

where $e = l$;



that is, vol. of triangular prism $= \frac{bh}{6} \times 3l$ solid units

$= \frac{bh}{2} \times l$ solid units

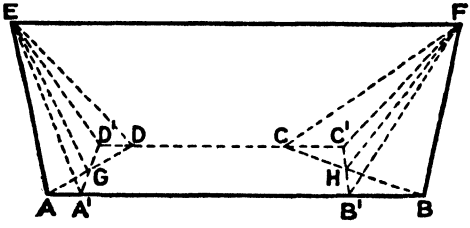
$= A_1 l$ solid units

where A_1 = number of square units in the cross-section.

This result has been already obtained in § 130.

148. The definition of a wedge may be extended so as to include the case when the base is *any trapezoid*, and not necessarily a rectangle.

It can be shown that the volume of such a wedge is the same as the volume of a wedge of equal edge and of equal height on a *rectangular* base, whose breadth is the same as the breadth of the



trapezoidal base, and whose length is equal to half the sum of the parallel sides of the trapezoidal base.

For consider the wedge $ABCDEF$, whose base $ABCD$ is a trapezoid.

Through the middle points G and H of AD and BC draw $A'D'$ and $B'C'$ perpendicular to AB, meeting DC or DC produced in D' and C'.

Join $A'E$, GE, $D'E$, $B'F$, HF, $C'F$.

Then, because pyramids on equal bases and of the same heights are equal in volume . . . . . § 139.

$\therefore$ vol. of pyramid $EAA'G$ = vol. of pyramid $EDD'G$

and vol. of pyramid $FBB'H$ = vol. of pyramid $FCC'H$

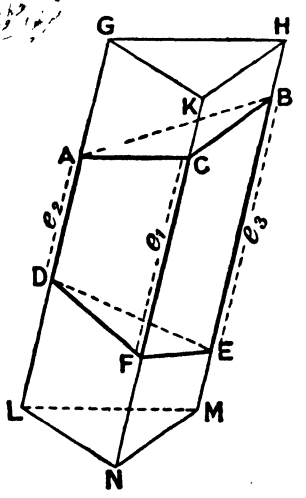
$\therefore$ volume of wedge $ABCDEF$ = volume of wedge $A'B'C'D'EF$

Hence—

Volume of wedge on trapezoidal base $= \frac{bh}{6}(2l + e)$ solid units

where l = number of linear units in half the sum of the parallel sides of the trapezoidal base.

149. A wedge on a trapezoidal base may be regarded as an oblique frustum of a triangular prism, that is, a part of a triangular prism included between two planes inclined to one another.



For example, the wedge $ABCDEF$ may be regarded as an oblique frustum of the triangular prism $GHKLMN$.

If the three parallel edges of the wedge $ABCDEF$ measure e_1, e_2, e_3 of the same linear unit respectively, we may make the following substitutions in the formula :—

$$V = \frac{bh}{6}(2l + e)$$

namely—

$$l = \frac{e_2 + e_3}{2}$$

$$e = e_1$$

The formula can now be written—

$$V = \frac{bh}{2} \cdot \frac{e_2 + e_3 + e_1}{3}$$

But $\frac{bh}{2}$ = area of cross-section of wedge

$= A_1$ square units

$$\therefore V = A_1 \cdot \frac{e_1 + e_2 + e_3}{3}$$

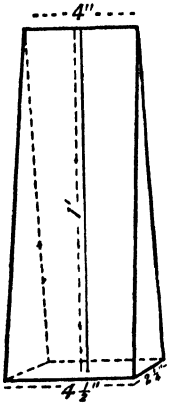
Hence rule—

Multiply the number of any square unit in the cross-section of a wedge (or oblique frustum of a triangular prism) by the number of the corresponding linear unit in the mean length of the parallel edges, then the product will give the number of the corresponding solid unit in the volume.

Or briefly—

Volume of wedge (or oblique frustum of triangular prism) $= \left\{\begin{matrix}\text{area of cross-section} \times \\ \text{mean length of par. edges}\end{matrix}\right.$

$$\therefore \quad \mathbf{V = A_1 \cdot \frac{e_1 + e_2 + e_3}{3}}$$



ILLUSTRATIVE EXAMPLES.

150. *Example* 1.—Find the weight of an iron wedge, the length and breadth of the base being $4\frac{1}{2}$ and $2\frac{1}{4}$ in., the length of the edge 4 in., and the height 1 ft., at the rate of 7788 oz. per cubic foot.

Vol. of wedge $= \frac{bh}{6}(2l + e)$ cub. in. § 146.

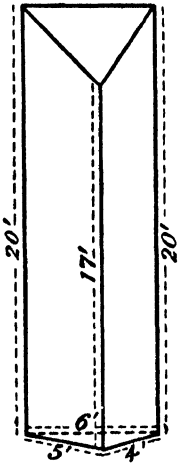
where $b = 2\frac{1}{4}$,
$h = 12$,
$l = 4\frac{1}{2}$,
$e = 4$;

$\therefore$ volume of wedge $= \frac{2\frac{1}{4} \times 12}{6}(2 \times 4\frac{1}{2} + 4)$ cub. in.

$= 4\frac{1}{2} \times 13$ cub. in.

$\therefore$ weight of wedge $= \frac{4\frac{1}{2} \times 13 \times 7788}{1728}$ oz.

$= 263\frac{21}{32}$ oz.



Example 2.—The breadths of the sides of a triangular prism are 4, 5, and 6 ft. The 6-ft. side is a rectangle 20 ft. long, but the other edge of the prism is only 17 ft. Find the volume.

Volume of prism $= A_1 \times \frac{e_1 + e_2 + e_3}{3}$ cub. ft. § 149.

where $A_1 = \sqrt{\frac{15}{2} \cdot (\frac{15}{2} - 6)(\frac{15}{2} - 5)(\frac{15}{2} - 4)}$. § 23.

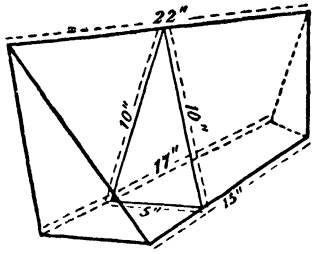
$= \frac{15\sqrt{7}}{4}$

$e_1 = 20$,
$e_2 = 20$,
$e_3 = 17$;

$\therefore$ volume of prism $= \frac{15\sqrt{7}}{4} \times \frac{20 + 20 + 17}{3}$ cub. ft.

$= 188{\cdot}509$ cub. ft.

Example 3.—The transverse section of a wedge is an isosceles triangle, the sides of which are double the base, and the perimeter of which is 25 in.; also the three parallel edges of the wedge are 15, 17, and 22 in. respectively: find its volume.



Let x in. = base of transverse section.

Then $x + 2x + 2x = 25$

$$x = 5$$

$\therefore$ volume of wedge $= \frac{bh}{6}(2l + e)$ cub. in. . § 148.

where $b = 5$,

$$h = \sqrt{10^2 - (\tfrac{5}{2})^2} = \frac{\sqrt{375}}{2} \quad \ldots\ldots\ldots\ldots \text{§ 16.}$$

$$l = \frac{17 + 15}{2},$$

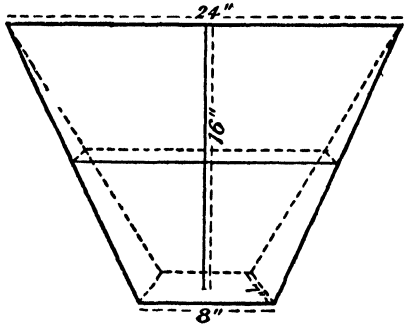
$$e = 22;$$

$$\therefore \text{volume of wedge} = \frac{5 \cdot \sqrt{375}}{2 \times 6}(2 \times 16 + 22) \text{ cub. in.}$$

$$= \frac{5\sqrt{375} \times 54}{12} \text{ cub. in.}$$

$$= 435{\cdot}7 \text{ cub. in.}$$

Example 4.—The edge of a wedge is 24 in., the length of its base 8 in., and the breadth 7 in.; the height of the wedge is 16 in. The wedge is divided into two parts by a plane parallel to the base midway between the edge and the base. Find the volume of each part.



Vol. of whole wedge $= \frac{bh}{6}(2l + e)$ cub. in. § 146.

where $b = 7$,
$h = 16$,
$l = 8$,
$e = 24$;

$$\therefore \text{volume of whole wedge} = \frac{7 \times 16}{6}(2 \times 8 + 24) \text{ cub. in.}$$

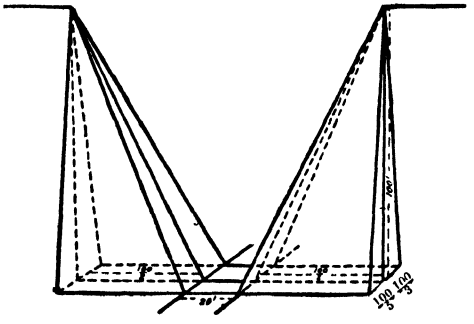
$$= 746\tfrac{2}{3} \text{ cub. in.}$$

Volume of smaller wedge cut off by the plane $= \frac{bh}{6}(2l + e)$ cub. in. § 146.

where $b = \frac{7}{2}$,
$h = \frac{16}{2}$,
$l = \frac{24 + 8}{2}$,
$e = 24$;

$$\therefore \text{volume of smaller wedge} = \frac{7 \times 8 \times 56}{2 \times 6} \text{ cub. in.}$$
$$= 261\tfrac{1}{3} \text{ cub. in.}$$
$$\therefore \text{volume of remaining part of whole wedge} = 485\tfrac{1}{3} \text{ cub. in.}$$

Example 5.—A cutting for a road is to be made through a ridge which has a slope of 5 to 1 on the one side, and of 3 to 1 on the other side. The highest point of the ridge is 100 ft. above the formation level, the width of the road 20 ft., and the slopes of the sides of the cutting 2 to 1. What will it cost at Rs.5 per 1000 cub. ft. ?



The cutting is seen by the figure to be a wedge of the following dimensions :—

$$\text{Breadth of base} = (\tfrac{100}{5} + \tfrac{100}{3}) \text{ ft.}$$
$$= \tfrac{160}{3} \text{ ft.}$$
$$\text{length of base} = 20 \text{ ft.}$$
$$\text{height} = 100 \text{ ft.}$$
$$\text{edge} = (20 + \tfrac{100}{2} + \tfrac{100}{2}) \text{ ft.}$$
$$= 120 \text{ ft.}$$

$$\therefore \text{cubical content of cutting} = \frac{bh}{6}(2l + e) \text{ cub. ft.} \quad . \quad \S\ 146.$$

where $b = \frac{160}{3}$,
$h = 100$,
$l = 20$,
$e = 120$;

$$\text{hence cost of cutting} = \frac{160 \times 100}{3 \times 6} \times (40 + 120) \times \tfrac{5}{1000} \text{ Rs.}$$
$$= \text{Rs.}711\ 1 \text{ anna } 9\tfrac{1}{3} \text{ pies}$$

*

Examples XXIV.

1. The edge of a wedge is 18 in., the length of the base is 15 in., and the breadth of the base is 9 in., the height of the wedge is 14 in. : find the volume.

2. The edge of a wedge is 2 ft. 9 in., the length of the base is 3 ft., and the breadth of the base is 1 ft. 3 in., the height of the wedge is 1 ft. 6 in. : find the volume.

3. The edge of a wedge is 1 ft. 8 in., the area of a section of the wedge made by a plane perpendicular to the edge is 1 sq. ft. : find the volume if the length of the base is 2 ft.

4. The section of a wedge made by a plane perpendicular to the edge is an equilateral triangle each side of which is 8 in. : find the volume if the edge of the wedge is 18 in. and the length of the base 21 in.

5. The base of a wedge is a square of side 12 in., the height of the wedge is 21 in., and the edge 27 in. : find the volume.

6. A wedge-shaped trench is 7 ft. wide at the top and 30 yds. long ; the length of the edge along the bottom is 26 yds., and the depth of the trench is 9 ft. : find the weight of the earth excavated if 1 cub. ft. of earth weigh 95 lbs.

7. Find the volume of the frustum of a prism whose cross-section is an equilateral triangle of side 3 ft., and the sum of whose three parallel edges is 12 ft.

Examination Questions—XXIV.

Wedges.

A. *Bombay University, Diploma of Agriculture : Second Exam.*

1. The length of the edge of a wedge is $5\frac{1}{2}$ in., the length of the base is 3 in., and its breadth is 2 in., the height of the wedge is 4 in. : find its volume.

2. Find the volume of a wedge, the length and breadth of the base being 5 ft. 4 in. and 9 in. respectively, the length of the edge being 3 ft. 6 in., and the height 2 ft. 4 in.

B. *Bombay University : L.C.E. Second Exam.*

3. A cutting for a canal is to be made through a uniform ridge in the shape of a double inclined plane, the altitude of the highest point of the ridge above the level of the bed of the canal being 16 metres, and the gradient being 9 in 41 at each side. The breadth of the canal bed being 5 metres, and the uniform slope of the banks 1 in 2, find the total quantity of earth excavated.

C. *Punjab University : First Exam. in Civil Engineering.*

4. State the formula for finding the volume of a wedge.

D. *Calcutta University : F.E. Exam.*

5. The edge of a wedge is 15 in., the length of the base is 24 in., and the breadth 7 in. ; the height is 22 in. The wedge is divided into a pyramid and a prism by a plane through one end of the edge parallel to the triangular face at the other end. Find the volume of each part.

E. *Sibpur Engineer Dept. : Annual Exam.*

6. A cylindrical vessel 1 ft. high, and the radius of whose base is 6 in., is full of water. A wedge whose edge is 7in., whose base is 5 in. long and 4 in. broad, and whose height is 6 in., is gently dipped into the water so that the water runs over ; it is then withdrawn. At what height in the vessel will the water now stand ?

F. *Sibpur Apprentice Dept.: Monthly Exam.*

7. The edge of a wedge is 21 in., the length of the base is 27 in., the area of a section of the wedge made by a plane perpendicular to the edge is 160 sq. in. : find the volume.

8. The edge of a wedge is 21 in., the length of the base is 15 in., and the breadth 9 in., the height of the wedge is 6 in. The wedge is divided into three parts of equal heights by planes parallel to the base. Find the volume of each part.

G. *Roorkee Engineer: Entrance.*

9. The edge of a wedge is 9 ft., the length of the base is 6 ft., and the breadth is 4 ft., the height of the wedge is $2\frac{1}{2}$ ft. : find the volume.

H. *Roorkee Upper Subordinate: Entrance.*

10. The edge of a wedge is 25 in., the length of the base is 22 in., a section of the wedge made by a plane perpendicular to the edge is an equilateral triangle, each side of which is 10 in. : find the volume.

I. *Staff College.*

11. *AE*, *BF*, *CG*, *DH*, are the vertical edges of a cubic foot of wood whose horizontal faces are *ABCD*, *EFGH*. In *AB* a point *M* is taken 7 in. from *A*, and in *AD* a point *N* 5 in. from *A*. A portion of the cube is cut away by a plane through *M*, *F*, *G*, and then a second portion by a plane through *N*, *H*, *G*. Find the volume of the three portions into which the cube is thus divided.

Frusta of Triangular Prisms.

Sibpur Apprentice Dept.: Monthly Exam.

12. The base of a prism is an equilateral triangle, each side of which is 4 in. : find the volume of the solid obtained by cutting off a piece of this prism so that the sum of the parallel edges is 42 in.

Additional Examination Question—XXIV.

13. The base of a wedge is a square, a side of which is 15 in., the edge is 24 in., and the height of the wedge is 24 in. : find the volume. (Roorkee Engineer: Final.)

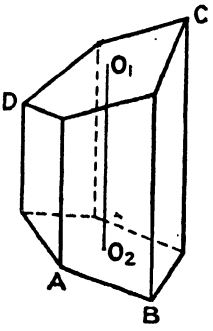
CHAPTER XXV.

ON OBLIQUE FRUSTA OF RIGHT REGULAR PRISMS, AND OBLIQUE FRUSTA OF RIGHT CIRCULAR CYLINDERS.

151. An *oblique frustum of a prism* is the part of a prism included between two planes inclined to one another.

The *length* of an oblique frustum of a prism is that portion of the axis of the prism which is intercepted by the two planes inclined to one another.

Thus O_1O_2 is the *length* of the oblique frustum $ABCD$.



PROPOSITION XXXVII.

152. *To find the volume of an oblique frustum of a right regular prism, having given the area of its cross-section and its length.*

Let $E_2A_2C_1$ be an oblique frustum of a right regular prism.

Let its cross-section $ABCDE$ measure A_1 of any square unit.

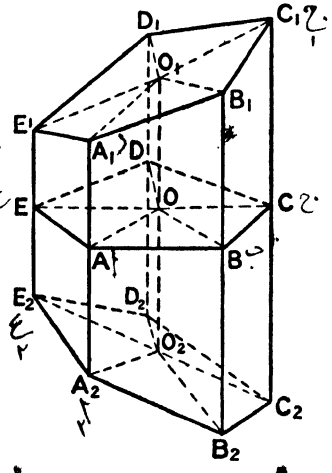
Let its length O_1O_2 measure l of the corresponding linear unit.

It is required to find the volume of $E_2A_2C_1$ in terms of A_1 and l.

Let A_1A_2, B_1B_2, C_1C_2, D_1D_2, E_1E_2 measure l_1, l_2, l_3, l_4, l_5 of the same linear unit respectively.

Now, the whole frustum can evidently be divided into five triangular frusta, namely—

$O_1A_2B_2$, $O_1B_2C_2$, $O_1C_2D_2$, $O_1D_2E_2$, $O_1E_2A_2$



and each of these frusta has for its cross-section a triangle,

which is the fifth part of the regular pentagon $ABCDE$, and which therefore measures $\frac{A_1}{5}$ square units.

Hence, if—

$$V_1 \; V_2 \; V_3 \; V_4 \; V_5$$

be the number of the corresponding solid unit in each of these five triangular frusta respectively, we have—

$$\left.\begin{aligned} V_1 &= \frac{A_1}{5} \times \frac{l_1 + l_2 + l}{3} \\ V_2 &= \frac{A_1}{5} \times \frac{l_2 + l_3 + l}{3} \\ V_3 &= \frac{A_1}{5} \times \frac{l_3 + l_4 + l}{3} \\ V_4 &= \frac{A_1}{5} \times \frac{l_4 + l_5 + l}{3} \\ V_5 &= \frac{A_1}{5} \times \frac{l_5 + l_1 + l}{3} \end{aligned}\right\} \quad \ldots\ldots \S\ 149.$$

$$\text{and volume of whole frustum} = (V_1 + V_2 + V_3 + V_4 + V_5) \text{ solid units}$$

$$= \frac{A_1}{5}\left\{\frac{2(l_1 + l_2 + l_3 + l_4 + l_5) + 5l}{3}\right\} \text{ solid units}$$

But l is evidently the mean of l_1, l_2, l_3, l_4, l_5; that is—

$$l_1 + l_2 + l_3 + l_4 + l_5 = 5l$$

$$\therefore \text{ volume of whole frustum} = \frac{A_1}{5}\left(\frac{2 \times 5l + 5l}{3}\right) \text{ solid units}$$

$$= A_1 l \text{ solid units}$$

Hence rule—

Multiply the number of any square unit in the cross-section of an oblique frustum of a right regular prism by the number of the corresponding linear unit in the length, then the product will give the number of the corresponding solid unit in the volume.

Or briefly—

Volume of oblique frustum of right regular prism = **cross-section × length**

$$\mathbf{V = A_1 l}$$

PARTICULAR CASE.

153. Oblique frustum of right circular cylinder.

A cylinder has been defined as the limiting case of a prism (§ 125).

In the same way, an oblique frustum of a cylinder may be defined as the limiting case of an oblique frustum of a prism.

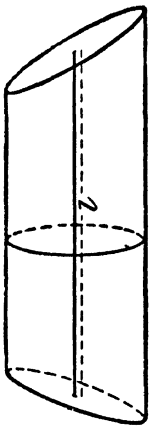
Hence the formula—

$$V = A_1 l$$

will determine the volume of the oblique frustum of a right circular cylinder, where A_1 is the number of any square unit in the cross-section of the frustum, and l is the number of the corresponding linear unit in the length.

If r linear units be the radius of the cross-section, this formula may be written—

$$V = \pi r^2 l$$



ILLUSTRATIVE EXAMPLES.

154. *Example* 1.—The base of a right prism is a regular octagon of side 2 ft. A frustum is obtained by cutting off a portion of this prism, so that the sum of the eight parallel edges is 64 ft. Find the volume of the frustum.

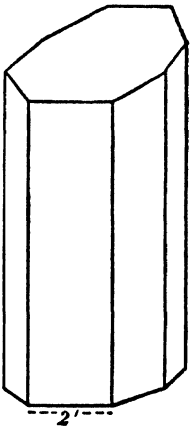
Volume of frustum $= A_1 l$ cub. ft. . § 152.

where $A_1 = 2 \times 2^2 \times (1 + \sqrt{2})$ . . . . § 45.

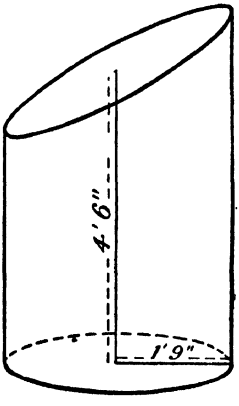
$l = \frac{64}{8} = 8$ . . . . . § 152.

$\therefore$ volume of frustum $= 64(1 + \sqrt{2})$ cub. ft.

$= 154{\cdot}509$ cub. ft.



Example 2. — The radius of the base of a right circular cylinder is 1 ft. 9 in.: find the volume of a frustum of this cylinder, if the length of the frustum is 4 ft. 6 in.



Volume of frustum $= A_1 l$ cub. ft. . § 153.

where $A_1 = \pi(\frac{7}{4})^2$ . . . . . § 71.

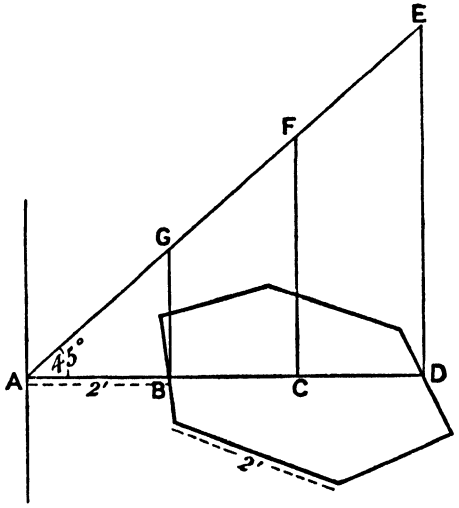
$l = \frac{9}{2}$.

$\therefore$ volume of frustum $= \frac{22}{7} \times \frac{49}{16} \times \frac{9}{2}$ cub. ft.

$= 43\frac{5}{16}$ cub. ft.

Example 3.—A regular hexagonal prism stands partially buried in a bank sloping at 45°, with one face parallel to the foot of the slope and two feet in rear of it: if the prism be 12 ft. in perimeter, find the volume of the buried portion.

Let $BDEG$ be a vertical mid-section of the buried portion of the prism by a plane at right angles to the foot of the slope.



Then—

$CF = AB + BC$ . . . . . Euc. I. 6.

But $AB = 2$ ft.

and BC = side of hexagon $\times \frac{\sqrt{3}}{2}$ . . . . . § 17.

$= \frac{12}{6} \times \frac{\sqrt{3}}{2}$ ft.

$= \sqrt{3}$ ft.

$\therefore$ CF $= (2 + \sqrt{3})$ ft.

Hence volume of buried portion $= A_1 \times l$ cub. ft. . . . . § 152.

where $A_1 = \frac{3 \cdot 2^2 \cdot \sqrt{3}}{2}$ . . . . . § 45.

$l = 2 + \sqrt{3}$;

hence volume of buried portion $= 6\sqrt{3}(2 + \sqrt{3})$ cub. ft.

$= 12\sqrt{3} + 18$ cub. ft.

$= 38{\cdot}7846$ cub. ft.

Examples—XXV.

(Take $\pi = \frac{22}{7}$.)

1. The base of a right prism is a regular hexagon of side 2 ft. A frustum is obtained by cutting off a portion of this prism so that the sum of the six parallel edges is 54 ft. Find the volume of the frustum.

2. The base of a right prism is a regular octagon of side 1 ft. A frustum is obtained by cutting off a portion of this prism so that the sum of the eight parallel edges is 56 feet. Find the volume of the frustum.

3. The cross-section of a prism is a regular dodecagon of side 2 in. A frustum is obtained by cutting off a portion of this prism so that the sum of the twelve parallel edges is 7 ft. Find the volume of the frustum.

4. The cross-section of a prism is a regular nonagon of side 1 ft. A frustum is obtained by cutting off a portion of this prism so that the sum of the nine parallel edges is 45 feet. Find the volume of the frustum.

5. The radius of the base of a right circular cylinder is 2 ft. 6 in. : find the volume of a frustum of this cylinder if the length of the frustum is 5 ft. 9 in.

6. The radius of the base of a right circular cylinder is 1 ft. 3 in. : find the volume of a frustum of this cylinder if the length of the frustum is 3 ft. 8 in.

Examination Questions—XXV.

(Take $\pi = \frac{22}{7}$.)

A. *Roorkee Engineer : Entrance.*

1. An octagonal stone prism stands at the foot of a sloping bank of grass which is inclined to the horizon at an angle of 45° ; the line of intersection of the slope with the ground is parallel to one face of the prism and 1 ft. in advance of it. If the prism be 8 ft. high and 12 ft. in perimeter, what proportion of its volume is above the bank ?

2. Find the weight in pounds of a marble column, in shape of a frustum of a cylinder, whose longest and shortest edges are 12 ft. 9 in. and 11 ft. 1 in., and whose radius of base is 1 ft. 3 in., if 1 cub. ft. of marble weighs 2716 ozs.

B. *Roorkee Upper Subordinate : Entrance.*

3. In a square pyramid whose height is equal to the side of the base, a circular hole is bored parallel to the plane of the base and to two sides, at a third of the height from the bottom : find the quantity of material cut out. The length of the base is 9 ft., the radius of the boring is 1 ft.

C. *Roorkee Engineer : Final.*

4. A cube of wood with an edge of 2 in. is pierced by a gimlet of $\frac{1}{2}$ an inch diameter, the point of the gimlet entering at the centre of one face and coming out at the centre of an adjacent face : find the volume of wood removed.

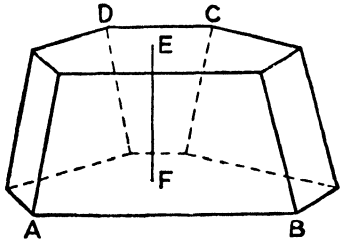
Additional Examination Question—XXV.

5. The radius of the base of a right circular cylinder is 2 ft. 6 in. : find the volume of the frustum of this cylinder, if the length of the frustum be 5 ft. 9 in. (Roorkee Engineer : Entrance.)

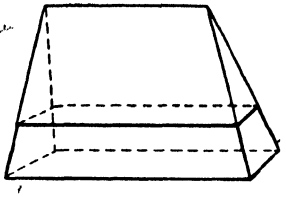
CHAPTER XXVI.

ON PRISMOIDS, FRUSTA OF WEDGES, FRUSTA OF PYRAMIDS, AND FRUSTA OF CONES.

155. A PRISMOID is a solid whose ends are rectilineal figures of the same number of sides, lying in parallel planes, and whose side surfaces are trapezoids.

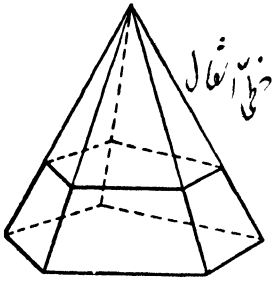


The perpendicular distance between the ends of a prismoid is called the *height* of the prismoid.

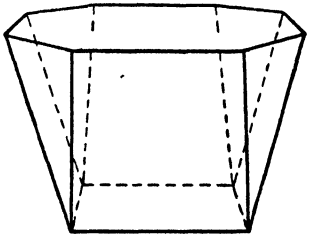


Thus EF is the height of the prismoid $ABCD$.

When the ends of a prismoid are *rectangles*, the prismoid may be seen to be a *frustum of a wedge*—that is, the part of a wedge included between the base and a plane parallel to the base.



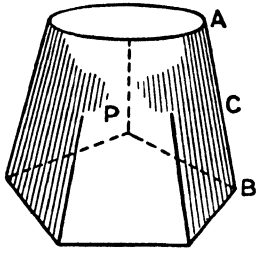
When the ends of a prismoid are *similar rectilineal figures similarly situated*, the prismoid may be seen to be a *frustum of a pyramid*—that is, the part of a pyramid included between the base and a plane parallel to the base.



156. The definition of a prismoid may be extended so as to include the case when any of the

trapezoidal sides become triangles through the vanishing of their smaller ends, as in the figure.

157. The definition of a prismoid may be still further extended so as to include the case when the ends are any two figures lying in parallel planes, its sides being straight, as in the figure.



By having its sides *straight* is meant that a straight line can be placed at any point on their surface, so as to coincide with the surface from end to end.

For example, through the point C on the side of the prismoid P, it is possible to draw a straight line AB coinciding with the surface of the prismoid from end to end.

Reservoirs, railway cuttings, and railway embankments often assume the form of a prismoid.

158. In proving the formula for the volume of a prismoid, we shall be content to take the special case of the frustum of a wedge, since the proof for the general case is considered too long and intricate for the purposes of this book.

PROPOSITION XXXVIII. مقطع ناقص

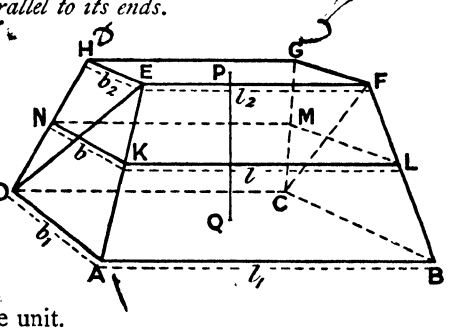
159. *To find the volume of a prismoid (frustum of a wedge), having given its height, the areas of its two ends, and the area of its mid-section by a plane parallel to its ends.*

Let $ABGH$ be a prismoid.

Let its height PQ measure h of any linear unit.

Let its ends $ABCD$ and $EFGH$ measure A_1 and A_2 of the corresponding square unit.

Let its mid-section $KLMN$ measure A of the corresponding square unit.



It is required to find the volume of $ABGH$ in terms of h, A_1, A_2, and A.

Divide the prismoid into two wedges, $ABCDEF$ and $EFGHCD$, by the plane passing through EF and DC.

Let AB, EF, KL, AD, EH, KN measure l_1, l_2, l, b_1, b_2, b of the same linear unit respectively.

Let the wedges $ABCDEF$ and $EFGHCD$ measure V_1 and V_2 of the corresponding solid unit respectively.

Then—

$$V_1 = \frac{b_1 h}{6}(2l_1 + l_2) \quad \text{§ 146.}$$

$$\text{and } V_2 = \frac{b_2 h}{6}(2l_2 + l_1) \quad \text{§ 146.}$$

$$\therefore V = V_1 + V_2 = \frac{h}{6}(2l_1b_1 + l_2b_1 + 2l_2b_2 + l_1b_2)$$

$$= \frac{h}{6}(l_1b_1 + l_1b_2 + l_2b_1 + l_2b_2 + l_1b_1 + l_2b_2)$$

$$= \frac{h}{6}\{(l_1 + l_2)(b_1 + b_2) + l_1b_1 + l_2b_2\}$$

$$= \frac{h}{6}\{(2l \times 2b) + l_1b_1 + l_2b_2\}$$

$$= \frac{h}{6}(4A + A_1 + A_2)$$

Hence rule—

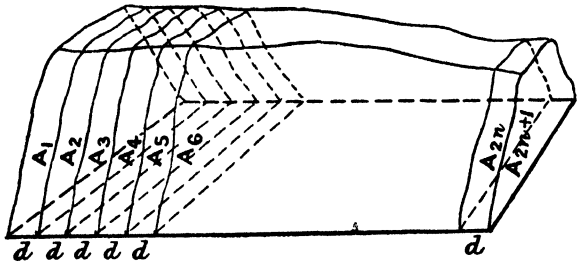
Add together the numbers of any square unit in the ends of a prismoid and four times the number of the same square unit in the section parallel to the ends and midway between them; multiply the sum by the number of the corresponding linear unit in the height, then one-sixth the product will give the number of the corresponding solid unit in the volume.

Or briefly—

$$\textbf{Volume of prismoid} = \frac{\textbf{height}}{6}(\textbf{sum of areas of ends} + 4 \times \textbf{area of mid-section})$$

$$\mathbf{V} = \frac{\mathbf{h}}{\mathbf{6}}(\mathbf{A_1} + \mathbf{A_2} + \mathbf{4A})$$

160. By an application of the prismoidal formula we can determine approximately the volume of any figure of which the



two opposite ends lie in parallel planes, but which is not, strictly speaking, a prismoid—embankments and railway cuttings, for example, where the surface of the ground is more or less uneven. For if the length of the embankment be divided into a number of equal parts by planes parallel to the ends (see figure), the earth between any two *alternate* planes may be regarded as a prismoid, and the transverse section by the intermediate plane will be the mid-section of this prismoid.

Hence its volume can be determined separately by means of the formula—

$$V = \frac{h}{6}(A_1 + A_2 + 4A)$$

Similarly the volumes of other parts of the earthwork can be determined, and the greater the number of parts the more accurate will be the result, in spite of the inequalities in the surface of the ground.

The student will now have no difficulty in proving the following formula for determining approximately the volume of any solid the opposite ends of which lie in parallel planes :—

$$V = \frac{d}{3}\{A_1 + A_{2n+1} + 2(A_3 + A_5 + \ldots + A_{2n-1}) + 4(A_2 + A_4 + \ldots + A_{2n})\}$$

where V = volume,

d = common distance between the parallel planes,

$2n$ = number of equal parts into which the length of the solid is divided by the parallel planes,

A_1, A_2, A_3 . . . A_{2n}, A_{2n+1} are the areas of the transverse sections of the solid made by the parallel planes taken in order.

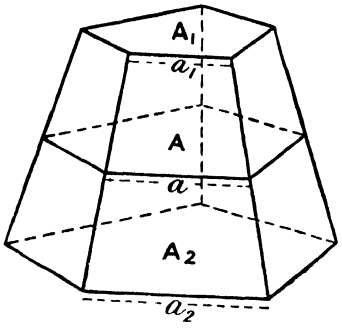
Notice the resemblance between this formula and the formula of Simpson's Rule (see Chapter XVI.).

161. We have proved that the formula—

$$V = \frac{h}{6}(A_1 + A_2 + 4A)$$

determines the volume of a prismoid when the prismoid assumes the special form of the frustum of a wedge. The same formula, however, is of much wider application.

Not only does it hold for all prismoids as defined in §§ 155, 156, 157, but it also determines the volumes of certain solids, which, however, cannot be defined in an elementary manner. Suffice it to say that these solids are called "prismoidal solids," and include, among others, the sphere, the segment of a sphere, and the zone of a sphere, the oblate spheroid and the prolate spheroid.



PARTICULAR CASES.

162. (1) Frustum of a pyramid.

Here the two ends and the mean section are similar figures similarly situated (§ 155).

If then, any corresponding sides of the top, bottom, and mean section measure a_1, a_2, and a of the same linear unit respectively, we have—

$$A_1 : A_2 : A = a_1^2 : a_2^2 : a^2 \quad \text{§ 104.}$$

$$\therefore \sqrt{A_1} : \sqrt{A_2} : \sqrt{A} = a_1 : a_2 : a$$

But $2a = a_1 + a_2$

$$\therefore 2\sqrt{A} = \sqrt{A_1} + \sqrt{A_2}$$

and squaring, $4A = A_1 + A_2 + 2\sqrt{A_1 A_2}$

Now, for all prismoids—

$$V = \frac{h}{6}(A_1 + A_2 + 4A). \quad . \quad . \quad . \quad . \quad \text{§ 161.}$$

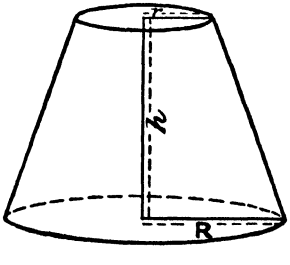
hence, for the frustum of a pyramid—

$$V = \frac{h}{6}\{A_1 + A_2 + (A_1 + A_2 + 2\sqrt{A_1 A_2})\}$$

$$= \frac{h}{6}(2A_1 + 2A_2 + 2\sqrt{A_1 A_2})$$

$$= \frac{h}{3}(A_1 + A_2 + \sqrt{A_1 A_2})$$

(2) Frustum of a cone.



A cone has been defined as the limiting case of a pyramid (§ 137).

In the same way, a frustum of a cone may be defined as the limiting case of a frustum of a pyramid.

Hence the formula—

$$V = \frac{h}{3}(A_1 + A_2 + \sqrt{A_1 A_2})$$

will determine the volume of a frustum of a cone.

In the case of a frustum of a *right circular* cone, this formula can be simplified by putting—

$$A_1 = \pi R^2$$

$$A_2 = \pi r^2$$

where R and r are the numbers of the same linear unit in the greater and lesser ends of the frustum respectively (see figure).

Hence, for a frustum of a right circular cone—

$$V = \frac{h}{3}(\pi R^2 + \pi r^2 + \sqrt{\pi^2 R^2 r^2})$$
$$= \frac{\pi h}{3}(R^2 + r^2 + Rr)$$

Note.—Rectangular solids, parallelopipeds, prisms, cylinders, pyramids, cones, and wedges may all be regarded as particular cases of the prismoid, and their volume-formulæ can readily be deduced from the formula—

$$V = \frac{h}{6}(A_1 + A_2 + 4A)$$

by introducing the special conditions in each case.

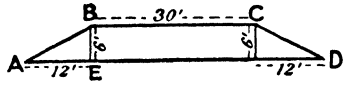
For example, in the case of the wedge—

$$A_1 = bl$$
$$A_2 = 0$$
$$A = \frac{l+e}{2} \times \frac{b}{2}$$
$$\therefore V = \frac{h}{6}\left(bl + 4 \times \frac{l+e}{2} \times \frac{b}{2}\right)$$
$$= \frac{bh}{6}(2l + e) \quad \ldots\ldots\ldots \text{§ 146.}$$

ILLUSTRATIVE EXAMPLES.

Prismoids.

163. *Example* 1.—Find the cubical content of an embankment 400 ft. long, the height at the ends being 6 ft. and 4 ft. respectively, the side slopes 1 : 2, and the breadth at the top 30 ft. throughout, the ends being vertical.



Let $ABCD$ represent the 6 ft. end of the embankment.

Then BC = 30 ft., and BE = 6 ft.

Therefore, since the side slopes are as 1 : 2—

$$AD = (30 + 2 \times 12) \text{ ft.}$$
$$= 54 \text{ ft.}$$

Similarly, we find that—

$$\text{Width of the embankment at the bottom of the 4-ft. end} = (30 + 2 \times 8) \text{ ft.}$$
$$= 46 \text{ ft.}$$

$\therefore$ cubical content of embankment $= \frac{h}{6}(A_1 + A_2 + 4A)$ cub. ft. § 161.

where $h = 400$,

$$A_1 = \frac{30 + 54}{2} \times 6 = 252 \quad \ldots \quad \S\ 39.$$

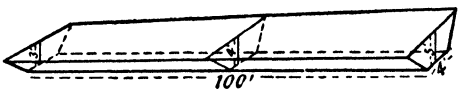
$$A_2 = \frac{30 + 46}{2} \times 4 = 152 \quad \ldots \quad \S\ 39.$$

$$A = \frac{30 + \frac{54 + 46}{2}}{2} \times \frac{6 + 4}{2} = 200 \quad \ldots \quad \S\ 39.$$

hence cubical content of embankment $= \frac{400}{6}(252 + 152 + 800)$ cub. ft.
$= 80{,}266{\cdot}6$ cub. ft.

Example 2.—Find the content of a drain 100 ft. long, whose depths of diggings at commencement, middle, and end are 3, 4, and 5 ft. respectively, side slopes 1 : 2, and bottom breadth 4 ft.

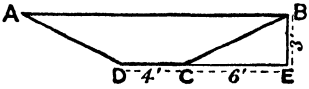
Since the middle depth 4 ft. is the mean between the two end depths, 3 ft. and 5 ft., the drain may be regarded as a prismoid having for its parallel ends and mid-section trapezoids whose areas A_1, A_2, and A sq. ft. can be easily ascertained.



To find A_1.

Let $ABCD$ represent the trapezoidal end whose area $= A_1$ sq. ft.

Draw BE perpendicular to DC produced.



Because the slope of BC is 1 : 2—

$\therefore CE = 2 \times BE = 6$ ft.

hence $AB = (4 + 12)$ ft. $= 16$ ft.

and $A_1 = \frac{16 + 4}{2} \times 3$. § 39.

$= 30$

Similarly, we can find—

$A_2 = 70$

$A = 48$

$$\therefore \text{volume of drain} = \frac{h}{6}(A_1 + A_2 + 4A) \text{ cub. ft.} \quad . \quad \S\ 161.$$

where $h = 100$,
$A_1 = 30$,
$A_2 = 70$,
$A = 48$;

hence volume of drain $= \frac{100}{6}(30 + 70 + 192)$ cub. ft.
$= 4866{\cdot}6$ cub. ft.

Example 3.—The sides of a tank are 30 and 20 ft. at the top, and 12 and 8 ft. at the bottom, and the depth is 8 ft. The tank is emptied in 3 hours by a pipe, the water in which runs at a uniform rate of 4 ft. a second. Find the diameter of the pipe.

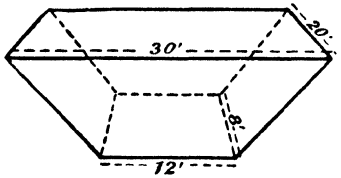
Cubical content of tank $= \frac{h}{6}(A_1 + A_2 + 4A)$ cub. ft. § 161.

where $h = 8$,

$A_1 = 30 \times 20 = 600$ . . § 8.

$A_2 = 12 \times 8 = 96$ . . . § 8.

$A = \frac{30+12}{2} \times \frac{20+8}{2} = 294$ § 8.



$\therefore$ cubical content of tank $= \frac{8}{6}(600+96+1176)$ cub. ft.

$= 2496$ cub. ft.

Let r in. = radius of pipe.

Then πr^2 sq. in. = cross-section of pipe . . . . § 71

and $\frac{\pi r^2 \times 4}{144}$ cub. ft. $= \frac{\pi r^2}{36}$ cub. ft.

= quantity of water held by 4 ft. of the pipe . . . . . § 131.

But this quantity of water flows out of the tank every second; therefore the tank will be emptied in $\left(2496 \div \frac{\pi r^2}{36}\right)$ seconds.

Hence $2496 \div \frac{\pi r^2}{36} = 3 \times 60 \times 60$

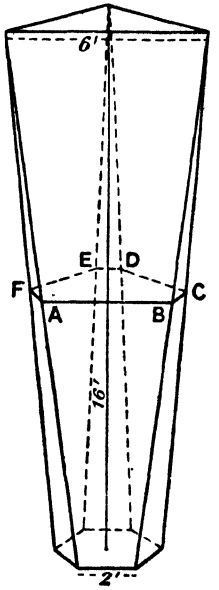
$\therefore r^2 = \frac{7\dot{2}8}{2\dot{1}5}$

$\therefore r^2 = 2{\cdot}64\dot{7}\dot{2}$

$\therefore r = 1{\cdot}627\ldots$

Hence the diameter of the pipe measures 3·25 in. nearly.

Example 4.—A prismoid has one end in the form of an equilateral triangle of side 6 ft., the other end in the form of a regular hexagon of side 2 ft., three sides of the hexagon being parallel to the three sides of the other end. The height is 16 ft. Find its volume.



The figure $ABCDEF$ may be taken to represent the mid-section of the prismoid, in which—

$$AB = CD = EF = 4 \text{ ft.}$$
$$\text{and } BC = DE = FA = 1 \text{ ft.}$$

But by producing AB, CD, EF both ways, the area of the figure $ABCDEF$ may be seen to be the difference between the area of an equilateral triangle of side 6 ft. and the sum of the areas of three equilateral triangles each of side 1 ft.

$\therefore$ volume of prismoid $= \frac{h}{6}(A_1+A_2+4A)$ cub. ft. § 161.

where $h = 16$,

$$A_1 = \frac{3 \times 2^2 \times \sqrt{3}}{2} = 6\sqrt{3} \quad \ldots \quad \S 45$$

$$A_2 = \frac{6^2\sqrt{3}}{4} = 9\sqrt{3} \quad \ldots \quad \S 24.$$

$$A = \left(\frac{6^2\sqrt{3}}{4} - \frac{3\sqrt{3}}{4}\right) = \frac{33\sqrt{3}}{4} \quad \ldots \quad \S 24.$$

Hence volume of prismoid $= \frac{16}{6}(6\sqrt{3} + 9\sqrt{3} + 33\sqrt{3})$ cub. ft.
$= 128\sqrt{3}$ cub. ft.
$= 221{\cdot}702$ cub. ft.

Example 5.—Find the cost at 5*s.* per 1000 cub. ft. of excavating the following trench: Length, 200 ft.; width of bottom, 15 ft.; longitudinal slope of bottom, 1 in 100; depth at upper end, 10 ft.; slope of sides, 1 in 10.

Since the trench is 200 ft. long, and the longitudinal slope of the bottom is 1 in 100, and the depth at the upper end is 10 ft.,

$\therefore$ the depth at the lower end is $(10 + \frac{1}{100} \times 200)$ ft. $= 12$ ft.

Now, the trench is evidently a prismoid, and its two ends are trapezoids whose bases, heights, and slope of sides we know, and so we can find their areas as in Example 2.

Hence—

$$\text{Volume of trench} = \frac{h}{6}(A_1 + A_2 + 4A) \text{ cub. ft.} \quad \ldots \quad \S 161.$$

where $h = 200$,

$A_1 = \frac{1}{2}\{15 + (15 + 2 \times 12 \times 10)\} \times 12 = 1620$ . . . . § 39.
$A_2 = \frac{1}{2}\{15 + (15 + 2 \times 10 \times 10)\} \times 10 = 1150$ . . . . § 39.
$A = \frac{1}{2}\{15 + (15 + 2 \times 11 \times 10)\} \times 11 = 1375$ . . . . § 39.

Hence volume of trench $= \frac{200}{6}(1620 + 1150 + 5500)$ cub. ft.

$$= \frac{200 \times 8270}{6} \text{ cub. ft.}$$

$\therefore$ cost of excavation $= \frac{200 \times 8270 \times 5}{6 \times 1000}$ shillings

$= £68\ 18s.\ 4d.$

Examples—XXVI. A.

1. The height of a prismoid is 10 ft., and the two ends are rectangles, the corresponding dimensions of which are 280 ft. by 250 ft. and 260 ft. by 190 ft.: find the volume.

2. The height of a prismoid is 3 ft. 4 in., and the two ends are rectangles, the corresponding dimensions of which are 5 ft. 6 in. by 4 ft. 8 in. and 3 ft. 2 in. by 2 ft. 10 in.: find the volume.

3. Find the weight of water required to fill a prismoidal cavity, the depth of which is 4 ft., and the top and bottom of which are rectangles, the corresponding dimensions of which are 24 ft. by 18 ft. and 22 ft. by 14 ft. Give the answer in tons.

4. An excavation 120 yds. long is uniformly 42 ft. wide at the bottom. It is 16 ft. deep at one end, and gradually increases to 20 ft. deep at the other,

and the upper widths at these ends are respectively 74 ft. and 86 ft. : find the cubic yards of excavation.

5. A prismoid is divided into two parts by a plane parallel to the ends and midway between them. If the ends of the prismoid are rectangles whose corresponding dimensions are 14 in. by 12 in. and 12 in. by 10 in., and if the height of the prismoid is 6 in., find the volume of each part.

6. The base of a prismoidal solid is an equilateral triangle, and the top a regular hexagon, three alternate sides of which are parallel to the sides of the base. The height of the solid is 10 in., the sides of the base 6 in., and those of the top 4 in. Find the volume.

7. How many gallons of water are required to fill a tank in the shape of a prismoid, the depth of which is 3 ft., and the top and bottom of which are rectangles whose corresponding dimensions are 40 ft. by 8 ft. and 32 ft. by 6 ft. ?

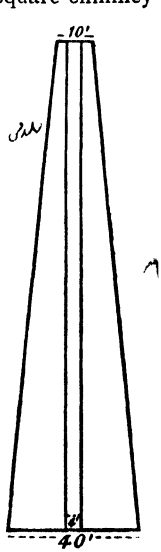
8. Find the cubic content of a piece of road embankment 300 ft. long, the longitudinal slope being regular, the height at the ends being 5 ft. and 3 ft. respectively, the side slopes 2 to 1, and the breadth at the top everywhere 26 ft., the ends being vertical.

9. Find the number of cubic yards of earth in a portion of a railway cutting 10 chains in length, the following numbers representing the areas in square yards of a series of transverse sections taken at intervals of one chain : 270, 256, 242, 238, 248, 250, 266, 272, 276, 272, 270.

ILLUSTRATIVE EXAMPLES.

Frusta of Pyramids and Frusta of Cones.

164. *Example* 1.—Find the cubical content of a square chimney shaft, 40 ft. broad at the base, and 10 ft. broad at the top, and having a batter of 1 in 10 ; the flue, which is circular, being 4 ft. in diameter throughout. ($\pi = 3{\cdot}1416$.)



Cubical content of shaft = cubical content of a frustum of a pyramid − cubical content of a cylinder.

Because the shaft has a batter of 1 in 10—

$\therefore$ height of shaft : 15 ft. = 10 : 1

$\therefore$ height of shaft = 150 ft.

$\therefore$ cubical content of frustum of pyramid $= \frac{h}{3}(A_1 + A_2 + \sqrt{A_1 A_2})$ cub. ft. . . § 162.

where $h = 150$,
$A_1 = 100$,
$A_2 = 1600$;

$\therefore$ cubical content of frustum of pyramid $= \frac{150}{3}(100 + 1600 + \sqrt{160000})$ cub. ft.
$= 105{,}000$ cub. ft.

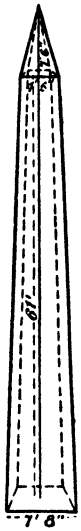
and cubical content of cylinder $= Ah$ cub. ft. . . . § 131.

where $A = \pi 2^2 = 4\pi$ . . . . . § 71.
$h = 150$;

$\therefore$ cubical content of cylinder = 600π cub. ft.

hence cubical content of shaft = (105,000 − 3·1416 × 600) cub. ft.
= 103,115·04 cub. ft.

Example 2.—Find the number of cubic feet in a frustum of a square pyramid 61 ft. high, 7 ft. 8 in. square at the base, and 4 ft. 6 in. square at the top, and capped at the top by a square pyramid 7 ft. 6 in. from the base to the apex.



Volume of frustum $= \frac{h}{3}(A_1+A_2+\sqrt{A_1A_2})$ cub. ft. § 162.

where $h = 61$,

$A_1 = 4\frac{1}{2} \times 4\frac{1}{2} = \frac{81}{4}$ . . . . . § 9.

$A_2 = 7\frac{2}{3} \times 7\frac{2}{3} = \frac{529}{9}$ . . . . . § 9.

$\therefore$ volume of frustum $= \frac{61}{3}(\frac{81}{4} + \frac{529}{9} + \frac{69}{2})$ cub. ft.

$= \frac{61}{3} \times \frac{4087}{36}$ cub. ft.

Volume of pyramid $= \frac{1}{3}Ah$ cub. ft. . . § 141.

where $A = 4\frac{1}{2} \times 4\frac{1}{2} = \frac{81}{4}$ . . . . . § 9.

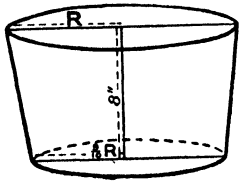
$h = 7\frac{1}{2}$;

$\therefore$ volume of pyramid $= \frac{1}{3} \times \frac{81}{4} \times \frac{15}{2}$ cub. ft.

hence volume of complete solid $= \left(\frac{61 \times 4087}{3 \times 36} + \frac{405}{8}\right)$ cub. ft.

$= 2359\frac{5}{216}$ cub. ft.

Example 3.—A bucket in the form of a frustum of a cone holds 4 galls. If the depth be 8 in., and the diameters at top and bottom be to each other as 10 to 9, find them.



Let R in. = radius of top of bucket.

Then—

Cubical content of bucket $= \frac{\pi h}{3}(R^2+r^2+Rr)$ cub. in. § 162.

where $h = 8$,

$r = \frac{9}{10} \times R$;

But the bucket holds 4 galls. or $277\frac{1}{4} \times 4$ cub. in. of water . . . . . § 112.

$\therefore \frac{\pi \times 8}{3}(R^2 + \frac{81}{100}R^2 + \frac{9}{10}R^2) = 277\frac{1}{4} \times 4$

$\therefore R^2 = \frac{1109 \times 3 \times 7 \times 100}{8 \times 22 \times 271}$

$= 48{\cdot}8279 \ldots$

$\therefore R = 6{\cdot}98 \ldots$

hence diameter at top $= 13{\cdot}9 \ldots$ in.

and diameter at bottom $= 12{\cdot}5 \ldots$ in.

Example 4.—The height of a conical frustum is 31 in., and the radius of one end 10 in.: determine the radius of the other end so that the frustum may be equal in volume to a right cylinder whose height is $\frac{1}{3}$ of an inch, and radius of base 62 in.

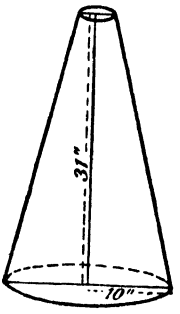
Let r in. = radius of the other end.

Then—

Volume of frustum $= \frac{\pi h}{3}(R^2 + r^2 + Rr)$ cub. in. . § 162.

where $h = 31$,
$R = 10$;

$$\therefore \text{volume of frustum} = \frac{\pi \times 31}{3}.(10^2 + r^2 + 10r) \text{ cub. in.}$$



Now, volume of a right cylinder whose height is $\frac{1}{3}$ of an inch, and radius of base 62 in. $= \pi(62)^2 . \frac{1}{3}$ cub. in. . . . . § 131.

Therefore, if the frustum and cylinder are equal in volume, we have—

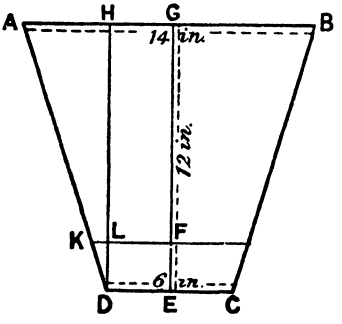
$$\frac{\pi \times 31}{3}(10^2 + r^2 + 10r) = \pi(62)^2 . \tfrac{1}{3}$$

or $10^2 + r^2 + 10r = 124$

or $r = 2$

Hence radius of the other end measures 2 in.

Example 5.—During a fall of rain, a bucket whose upper and lower interior diameters are 14 and 6 in., and depth 12 in., was placed on a flat surface, and after 20 minutes' exposure the depth of water in the bucket was found to be 2 in.: what was the rainfall per hour?



Let the figure $ABCD$ represent a vertical mid-section of the bucket after the rain has fallen.

Then—

$AG = 7$ in.
$DE = 3$ „
$EG = 12$ „
$EF = 2$ „

Through D draw DH perpendicular to AB.

By similar triangles—

$$KL : AH = DL : DH \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{§ 66.}$$

or $KL : 4 \text{ in.} = 2 : 12$

$\therefore KL = \frac{2}{3}$ in.

hence volume of water in bucket $= \frac{\pi h}{3}(R^2 + r^2 + Rr)$ cub. in.. § 162.

where $R = 3\frac{2}{3}$,
$r = 3$,
$h = 2$;

hence volume of water in bucket $= \frac{2\pi}{3}(\frac{121}{9} + 9 + 11)$ cub. in.

$$= \frac{602\pi}{27} \text{ cub. in.}$$

Now, consider a right circular cylinder whose cross-section is the same as the *mouth* of the bucket.

Such a cylinder will obviously admit the same quantity of rain-water in any given time as the bucket.

Hence, if h in. = depth of water in such a cylinder after 20 minutes' rainfall, we have—

$$\pi 7^2 h = \frac{602\pi}{27} \quad \ldots\ldots\ldots \S\ 131.$$

$$h = \frac{602}{27 \times 49}$$

or, depth after an hour's rainfall $= \frac{602 \times 3}{27 \times 49}$ in. $= \frac{86}{63}$ in.

But the depth of water in any cylinder after an hour's rainfall = rainfall per hour;

$\therefore$ rainfall per hour = 1·365079 in.

Examples—XXVI. B.

(Take $\pi = \frac{22}{7}$.)

Find the volumes of the following frusta of pyramids, having—

1. Areas of ends 8 sq. ft. and 5 sq. ft.; height 3 ft.

2. Areas of ends 12 sq. in. and 8 sq. in.; height 6 in.

3. Areas of ends 3 sq. ft. 72 sq. in. and 2 sq. ft. 28 sq. in.; height 2 ft. 6 in.

4. Areas of ends 4·85 sq. ft. and 3·15 sq. ft.; height 2·5 ft.

5. Find the volume of the frustum of a pyramid, the ends being rectangles whose corresponding dimensions are 36 in. by 24 in. and 27 in. by 18 in., and the height 15 in.

6. The ends of a frustum of a pyramid are equilateral triangles of sides 5 ft. and 7 ft. respectively; the height of the frustum is 4 ft.: find its volume.

Find the volumes of the following frusta of right circular cones, having—

7. Radii of ends 5 ft. and 6 ft.; height 4 ft. 6 in.

8. Radii of ends 3 ft. 8 in. and 2 ft. 4 in.; height 2 ft.

9. Radii of ends 6·75 ft. and 4·25 ft.; height 3·5 ft.

10. Radii of ends 3 yds. 2 ft. and 2 yds. 1 ft.; height 1 yd. 2 ft. 9 in.

11. The circumference of one end of a frustum of a right circular cone is 48 in., and of the other end 34 in.; and the height of the frustum is 10 in.: find the volume.

12. The ends of a frustum of a pyramid are squares, the lengths of the sides being 6 in. and 8 in. respectively; the height of the frustum is 4 in. The frustum is cut into two parts by a plane parallel to the two ends and midway between them: find the volume of each part.

13. The radii of the ends of a frustum of a right circular cone are 2 ft. 9 in. and 2 ft. 3 in. respectively; the slant height is 10 in.: find the volume.

14. The ends of a frustum of a pyramid are regular octagons whose sides measure 3 ft. and 5 ft. respectively · the height is 4 ft.: find the volume.

Examination Questions—XXVI.

(Take $\pi = \frac{22}{7}$.)

Prismoids.

A. *Allahabad University: Intermediate.*

1. A reservoir with slanting sides, whose base is 50 ft. by 40 ft., and top 75 ft. by 60 ft., is 15 ft. in perpendicular height: find the number of gallons it will hold.

B. *Bombay University, Diploma of Agriculture: Second Exam.*

2. The top and bottom of a reservoir in the shape of a prismoid are rectangles, the dimensions of the top being 200 ft. by 150 ft., and of the bottom 160 ft. by 130 ft.; its uniform depth is 12 ft.: find the cost of excavation at 1*s.* 6*d.* per cubic yard.

3. The length and breadth of a reservoir in the shape of a prismoid are 140 ft. and 80 ft. respectively; the length and the breadth of the bottom are 100 ft. and 60 ft. respectively, and the depth is 12 ft.: how many cubic feet of earth were dug out?

4. A piece of timber is 1 ft. 2 in. broad and 10 in. thick at one end, and 1 ft. 6 in. broad and 1 ft. thick at the other end, and 14 ft. long: find its volume.

5. A watercourse, 5 ft. wide at the bottom, 3 ft. deep at the upper end, and having a fall of 1 ft. in 320 yds., is to be cut in a straight line on level ground: if the sides are to slope 1 in 1, find the number of cubic yards of earth to be excavated in the first mile.

6. Enunciate the *prismoidal formula*, and apply it to find the volume of the following embankment: Length = 100 ft., end heights = 10 ft. and 4 ft., width of top = 5 ft., total slope of sides = 6 in 1.

C. *Bombay University: L.C.E. Second Exam.*

7. Find the quantity of earth excavated from a railway cutting made through ground which before disturbance was a uniform inclined plane running in the same direction as the rails, the length of the cutting being 100 yds., the breadth at the bottom 12 yds., the breadth at the top at one end being 45 yds. and at the other 25 yds., and the depths of these ends being 15 yds. and 7 yds. respectively.

8. The top widths of a railway cutting are 120 and 90 ft., their respective depths 30 ft. and 20 ft., the bottom width 30 ft., and the length of the cutting 66 yds.: find the contents in cubic yards.

9. A haystack 11½ ft. high has an oblong base 20 ft. long and 8 ft. broad, the sides of the rectangular horizontal section 9 ft. from the ground through the eaves are 22 ft. and 8·8 ft., and the part above the eaves forms a triangular prism 22 ft. long. If 10 cub. ft. of hay weigh 1 cwt., how many tons does the whole stack weigh?

10. Find the volumes of the wedge and prismoid into which a frustum of a pyramid is cut by a plane passing through one end of its base and cutting off a portion of the top 15 in. distant from its corresponding end, the length and breadth of the base being 45 and 30 in. respectively, those at the top being 36 and 24 in. respectively, and the height 40 in.

11. An excavation 858 ft. long is uniformly 50 ft. wide at the bottom; it is 18 ft. deep at one end, and gradually increases to 20 ft. deep at the other,

and the upper width of these ends are respectively 104 and 110 ft.: find the number of cubic yards in the excavation.

D. *Madras University: B.E. Exam.*

12. How many cubic feet of earth will be required to make a level embankment 1500 ft. long, 17 ft. and 12 ft. deep at the ends, 20 ft. wide on top, with side slopes of $1\frac{1}{2}$ to 1?

E. *Sibpur Engineer Dept.: Annual Exam.*

13. State and explain the prismoidal formula. A tank is 436 ft. by 325 ft. at the top, 376 ft. by 285 ft. at the bottom, and 10 ft. deep: how many cubic feet of water will be required to fill it three-quarters full, if there is in the middle a circular tower of 27 ft. diameter?

F. *Sibpur Apprentice Dept.: Monthly Exam.*

14. Show how to deduce the formula for finding the volume of a right circular cone from the prismoidal formula.

15. An earthen dam 300 ft. long has the two ends vertical; one is 20 ft. and the other 40 ft. in height. The bottom of the dam slopes uniformly from one end to the other; the top of the dam is horizontal and 20 ft. wide; side slope on one side is 2 horizontal to 1 vertical, and on the other is 4 horizontal to 1 vertical. Calculate the earthwork in the dam.

16. The ends of a prismoid are rectangles, the corresponding dimensions of which are 12 ft. by 10 ft. and 8 ft. by 6 ft.; the height of the prismoid is 4 ft. The prismoid is divided into two parts by a plane parallel to the ends and midway between them: find the volume of each part.

G. *Sibpur Apprentice Dept.: Annual Exam.*

17. A road is constructed along the greatest slope of a plane country. Top width of the road is 20 ft., side slopes are 2 horizontal to 1 vertical. Find the quantity of earthwork of a portion of the road 4 chains in length. Heights of embankment at the beginning and the end of that portion of the road are respectively 10 ft. and 20 ft.

18. Find the capacity of a coal-waggon the top of which measures 6 ft. 9 in. in length by 4 ft. 6 in. in breadth, the bottom 3 ft. 6 in. by 2 ft. 6 in., and the depth 4 ft.

H. *Sibpur Apprentice Dept.: Final Exam.*

19. A railway embankment is half a mile long, and has a uniform width of 30 ft. at the top. At one end it is 25 ft. high, and gradually decreases to the other end to 12 ft. high; the widths at the base at the ends are 120 ft. and 80 ft. respectively. Find the cost of making the embankment at Rs.5 per 1000 cub. ft.

I. *Roorkee Engineer: Entrance.*

20. A prismoid has one end in the form of an equilateral triangle of side 2 ft., the other end in the form of a regular hexagon of side 1 ft., three sides of the hexagon being parallel to the three sides of the other end; the height is 3 ft.: find its volume.

J. *Roorkee Upper Subord.: Entrance.*

21. Find the volume of a coal-waggon, the depth of which is 47 in.; the top and bottom are rectangles, the corresponding dimensions of which are 81 in. by 54 in., and 42 in. by 30 in.

22. Find the capacity of a trough in the form of a prismoid, its bottom being 48 in. long and 40 in. broad, its top 5 ft. long and 4 ft. broad, and the depth 3 ft.

K. *Superior Accounts.*

23. An embankment is made upon a slope of 1 in 10; the top of the embankment is horizontal, and its section is everywhere a trapezoid. The greatest height above the slope is 57 ft., the breadth of the top 26 ft., and the slopes of the sides 1 in 1. Find the number of cubic yards in a length of 160 yds. of the embankment.

Frusta of Pyramids.

A. *Allahabad University: Intermediate.*

24. The ends of a frustum of a pyramid are squares, the lengths of the sides being 20 ft. and 30 ft. respectively. The length of the straight line which joins the middle point of any side of one end with the middle point of the corresponding side of the other end is 13 feet. Find the volume.

B. *Madras University: B.E. Exam.*

25. The ends of a frustum of a pyramid are hexagons with sides of 6 and 4 ft. respectively; the slant height is 10 ft.: find the volume.

26. An unfinished spire is octagonal, and measures 80 ft. round the base. At a height of 55 ft. above the base, where the work was stopped, it measures 30 ft. round. How many cubic feet of masonry does it contain?

C. *Calcutta University: F.E. Exam.*

27. Find the volume of the frustum of a right pyramid on a regular base of n sides.

28. The areas of the ends of a frustum of a right pyramid being E_1 and E_2, and the thickness k, find the volume in the form

$$\frac{k}{3}(E_1 + \sqrt{E_1 E_2} + E_2)$$

D. *Sibpur Apprentice Dept.: Annual Exam.*

29. The bottom of a tank is a square whose area is 1 acre; its depth is 10 ft., and the side slopes are 1½ to 1: find the number of cubic feet of water it would contain.

E. *Sibpur Apprentice Dept.: Monthly Exam.*

30. The ends of a frustum of a pyramid are right-angled triangles; the sides containing the right angle of the one end are 2 ft. and 3 ft.; the smallest side of the other end is 8 ft.; the height of the frustum is 7 ft.: find the volume.

F. *Sibpur Apprentice Dept.: Final Exam.*

31. The ends of a frustum of a pyramid are equilateral triangles, the lengths of the sides being 6 ft. and 7 ft. respectively, and the length of the slant edge of the frustum is 9 ft.: find the volume.

G. *Roo[illegible]pper Subord.: Entrance.*

32. A tank in the shape of the frustum of an inverted right square pyramid, length of side at bottom 40 ft., and at ground level 120 ft. (the height of

pyramid if complete being equal to the side of the base), is to be lined with masonry 2 ft. thick : find the cost of the masonry at Rs.2 per cubic foot.

33. The base of a prismoidal solid is a square, and the top a regular octagon, four alternate sides of which are parallel to the sides of the base. The altitude of the solid is 16 ft., the sides of the base $3\frac{1}{2}$ ft., and those of the top 1 ft. : find its volume.

34. What is the solidity of a frustum of a regular hexagonal pyramid, the sides of the ends being 4 and 6 ft., and its length 24 ft. ?

H. *Roorkee Engineer : Final.*

35. The height of a frustum of a pyramid is 12·5 in. ; its ends are octagons whose sides are 4 and 2 in. respectively : find the volume of the frustum.

Frusta of Cones.

A. *Bombay University, Diploma of Agriculture : Second Exam.*

36. A cask, in the form of two conic frusta, joined at the bases, has the head diameter 20 in., the bung diameter 25 in., and the length 3 ft. 4 in. : find its capacity in gallons. ($277\frac{1}{4}$ cub. in. = 1 gall.)

37. The circumference of the base of a haystack in the form of a conical frustum surmounted by a cone is 40, the circumference at the eaves is 60, the perpendicular height of the frustum is 15, and that of the cone 16 ft. : how many solid yards does the stack contain ?

B. *Bombay University : L.C.E. Second Exam.*

38. A vessel is in the form of a frustum of a right circular cone, of which the bottom diameter is 32 in. The diagonals of a section by a plane passing through the axis cut each other so that the segments of each are 30 and 20 in. Determine the volume of the vessel.

C. *Punjab University : First Exam. in Civil Engineering.*

39. What is the volume of the frustum of a right cone, the area of the two circular ends being 1256·64 in. and 78·54 in. respectively, 30 in. being the height of the cone before it was truncated ?

D. *Madras University : B.E. Exam.*

40. A silver tumbler is of the shape of a truncated cone. Upper diameter inside 6 in., lower diameter 3 in., height 6 in., thickness of metal $\frac{1}{8}$ in. Find the weight. (Specific gravity, 11·00.)

41. A circular well is 17 ft. 6 in. in diameter, and 33 ft. deep : find the quantity of masonry lining, which is 2 ft. thick at the top and 4 ft. 9 in. at the bottom, the batter being on the rear.

42. State the rule for finding the volume of a frustum of a right circular cone.

43. The depth of a pail in the form of a frustum of a cone is 10 in., its diameter at the mouth 12 in., and its diameter at the bottom 9 in. : find how often it can be filled from a tank containing 2000 galls. of water. (A gallon = 277·274 cub. in.)

E. *Calcutta University : .E. Exam.*

44. In order to drain an acre of land, a tank is dug in the form of a frustum of a cone, the radius of the surface section being 30 yds., and of the bottom

20 yds., and the depth of the tank 15 ft. Assuming that two-fifths of the rainfall does not penetrate the soil, that there is no drainage from the subsoil, and no evaporation, find what the average daily rainfall has been if, after two months, the tank is two-thirds full. (One month = 30 days.)

F. *Sibpur Engineer Dept.: Annual.*

45. The frustum of a right circular cone is made of iron. Its height is 15 in.; the diameter of its smaller face is 9 in., and of its larger 16 in. Two conical holes are bored, one from each plane face. The diameters of the bases of these holes are each half the diameter of the corresponding face; their axes coincide with the axis of the frustum, and their vertices meet at the middle point of the axis. The holes are filled with lead. Find the weight of the whole, if the specific gravity of iron is 7·8, and of lead 11·4.

46. An iron right circular cone, 10 in. high, and whose semi-vertical angle is 30°, is cut into two at the mid-point of its height by a plane parallel to the base. The frustum so obtained is drawn into wire, whose diameter is $\frac{1}{10}$ in.: find the length of the wire.

G. *Sibpur Apprentice Dept.: Monthly.*

47. The shaft of Pompey's pillar is a single stone of granite. The height is 90 ft.; the diameter at one end is 9 ft., and at the other end 7 ft. 6 in. Find the volume.

48. The slant side of the frustum of a right circular cone is 5 ft., and the radii of the ends are 7 ft. and 10 ft.: find the volume.

H. *Sibpur Apprentice Dept.: Final Exam.*

49. Verify by calculating two or three cases the following statement: A right circular cone is divided into a cone and a frustum of a cone, and the frustum is trimmed just enough to reduce it to a right circular cylinder. If the height of the frustum is one-third of the height of the original cone, the volume of the cylinder is greater than in any other case, and is four-ninths of the original cone.

50. The height of the frustum of a cone is 7 ft., and the radii of the two ends are 4 ft. and 5 ft. respectively; the frustum is cut into two pieces by a plane parallel to the ends, and distant 3·884 ft. from the smaller end. Show that the two pieces are of equal volume.

I. *Roorkee Engineer: Entrance.*

51. A mast is 30 in. in diameter at bottom and 15 in. at top: if the mast contains 137½ cub. ft. of wood, find its height in feet.

52. A bucket is in shape a conical frustum (height = 9 in., diameters of top and bottom surface = 10 in. and 7½ in. respectively): find how much lower the water will stand in a well, whose diameter is 5 ft., after the bucket has been filled twenty-four times.

53. During a fall of rain a common bucket 12 in. deep was placed out on a level terrace, and at the end of one hour it was found that the water stood in the bucket at a perpendicular height of 4 in. The diameter of the bucket at the mouth and bottom was 9 inches and 3 in. respectively. Find the rate per hour at which the rain was falling.

54. A piece of marble in the form of a frustum of a cone has its end diameters 1½ and 4 ft., and its slant side is 8 ft.: what will it cost at 12*s.* the cubic foot?

55. A coppersmith proposes to make a flat-bottomed kettle, of the form of a conic frustum, to contain 13·8827 galls.; the depth of the kettle to be 1 ft., and the diameters of the top and bottom to be in the ratio of 5 to 3: what are the diameters?

56. If a cask, which is two equal conic frusta joined together at the bases, has its bung diameter 36 in., its head diameter 20 in., and its length 40 in., how many imperial gallons will it hold?

57. A balcony is supported by six granite columns of the following dimensions; the diameters of each at top and bottom are 2 and 2½ ft. respectively, and length 20 ft. Taking the rate at Rs.2 per cubic foot, what would be their total cost?

J. *Roorkee Upper Subord.: Entrance.*

58. A log of wood is in the form of a frustum of a cone; the diameter of the larger end is 16 in., and of the smaller end 12 in.; perpendicular height 9 ft.: what is its value at R. 1 8 ans. per cubic foot?

59. A frustum of a pyramid on a square base is trimmed just enough to reduce it to the frustum of a cone: show that rather more than one-fifth of the original volume is removed.

60. The radii of the ends of a frustum are 15 and 24 ft., and the slant height is 12 ft.: find the volume.

61. The lower portion of a haystack is an inverted conic frustum, and the upper part a cone. The greatest height is 25 ft., the greatest circumference is 54 ft., the height of the frustum 15 ft., and the diameter of the base 15 ft. Find the content in cubic yards.

K. *Roorkee Upper Subord.: Monthly.*

62. A bucket is in the form of a frustum of a cone. The diameter at the bottom is 1 ft., and at the top 1 ft. 3 in.; the depth is 1 ft. 6 in. Find to the nearest pound how much more the bucket weighs when full of water than when empty.

63. Find the number of cubic feet of masonry in a chimney shaft of the following dimensions: diameter of the base 30 ft. and of the top 6 ft., diameter of the flue at the base 3 ft. and at the top 2 ft. The outer face of the chimney is built with a batter of 1 in 20.

64. A well in the form of a cylinder, 4 ft. wide and 12 ft. deep, is emptied by a bucket 21 in. wide at the top, 18 in. wide at the bottom, and 15 in. deep: how many times must the bucket be lowered to empty the well, supposing that on the average when withdrawn it is only eight-ninths full?

L. *Roorkee Engineer: Final.*

65. A circular chimney shaft stands upon a solid circular foundation 30 ft. in diameter and 10 ft. deep. The measurements of the shaft are—

(i.) Perpendicular height, 200 ft.
(ii.) External diameter at base, 28 ft.
(iii.) External diameter at top, 8 ft.
(iv.) Diameter of flue, 4 ft.

Find the quantity of masonry in the whole.

66. During a fall of rain, a bucket whose upper and lower interior diameters are 15 and 8 in., and depth 13 in., was placed out on a flat surface, and after thirty minutes' exposure the depth of water in the bucket was found to be 4 in.: what was the rainfall per hour?

Additional Examination Question—XXVI.

67. On a hill with a slope of 1 to 4, a mound with a level top in the form of a square of 28 ft. side is to be constructed. One pair of sides of the square to run across, and the other in the direction of the slope. Sides of mound to slope at an angle of 45°. One side of the square to be 5 ft., and the opposite side to be 12 ft. above the original ground surface. Find the volume of the mound in cubic feet. (Roorkee Engineer: Entrance.)

CHAPTER XXVII.

ON SPHERES, SPHERICAL SHELLS, AND SPHEROIDS.

165. A *sphere* is a solid bounded by one surface, and is such that all straight lines drawn from a certain point within the solid to the bounding surface are equal to one another.

This point is called the *centre* of the sphere.

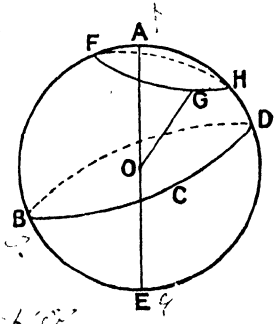
A *radius* of a sphere is a straight line drawn from the centre to the bounding surface.

A *diameter* of a sphere is a straight line drawn through the centre and terminated both ways by the bounding surface.

The section of a sphere by any plane is a circle.

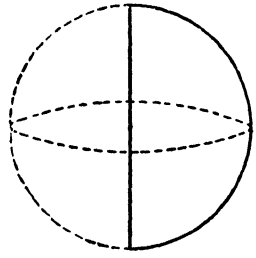
If the cutting plane pass through the centre of the sphere, the section is called a *great circle*.

If the cutting plane does not pass through the centre of the sphere, the section is called a *small circle*.



Thus, in the sphere $ABED$—

O is the centre,
OG is a radius,
AE is a diameter,
BCD is a great circle,
FGH is a small circle.



A sphere may be seen to be generated by the revolution of a semicircle about its diameter (see figure).

A tennis ball may be taken as a familiar example of a sphere.

PROPOSITION XXXIX.

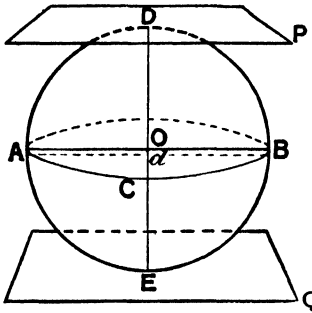
166. *To find the volume of a sphere having given its diameter.*

Let $ADBE$ be a sphere.

Let its diameter AB measure d of any linear unit.

It is required to find the volume of $ADBE$ in terms of d.

Since a sphere is a prismoidal solid (§ 161), therefore the formula—



$$V = \frac{h}{6}(A_1 + A_2 + 4A)$$

will determine its volume. § 161.

Now, if we regard the points of contact between the sphere and the two parallel tangential planes P and Q as two parallel ends of the sphere, then the circle ABC may be regarded as the section midway between these parallel ends, and the diameter DE will be the distance between the parallel ends.

Hence, in the formula—

$$V = \frac{h}{6}(A_1 + A_2 + 4A)$$

we may write—

$$h = d$$
$$A_1 = 0$$
$$A_2 = 0$$
$$A = \frac{\pi d^2}{4} \quad \ldots\ldots\ldots\ldots \text{§ 71.}$$

$$\therefore \text{volume of sphere} = \frac{d}{6}\left(0 + 0 + 4 \cdot \frac{\pi d^2}{4}\right) \text{solid units}$$

$$= \frac{\pi d^3}{6} \text{ solid units}$$

Hence rule—

Multiply the cube of the number of any linear unit in the diameter of a sphere by π, *then one-sixth the product will give the number of the corresponding solid unit in the volume.*

Or briefly—

$$\textbf{Volume of sphere} = \frac{\pi}{6} \times (\textbf{diameter})^3$$

$$\mathbf{V} = \frac{\pi \mathbf{d}^3}{6} \quad \ldots\ldots\ldots \text{(i.)}$$

$$\therefore \mathbf{d} = \sqrt[3]{\frac{6\mathbf{V}}{\pi}} \quad \ldots\ldots\ldots \text{(ii.)}$$

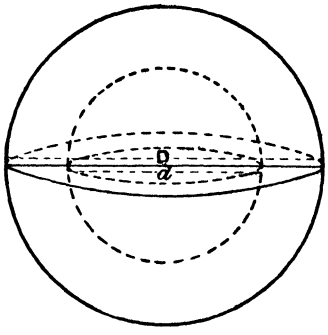
167. It is easy to deduce a formula for the volume of a

spherical shell in terms of the diameters of the two concentric spheres. For the volume of the shell is equal to the difference between the volumes of the two concentric spheres.

That is—

$$V = \frac{\pi D^3}{6} - \frac{\pi d^3}{6} \quad \text{§ 166.}$$

$$= \frac{\pi}{6}(D^3 - d^3)$$



where D and d are the diameters of the two spheres, and V is the volume of the shell.

$$\therefore\ V = \frac{\pi}{6}(D - d)(D^2 + Dd + d^2)$$

$$= \frac{\pi}{6} \cdot D^2 \cdot (D - d)\left(1 + \frac{d}{D} + \frac{d^2}{D^2}\right)$$

Now, suppose the thickness of the shell to be small compared with the diameter D; then $\frac{d}{D}$ is nearly equal to unity, and—

$$V = \frac{\pi}{6} \cdot D^2(D - d) \cdot 3 \text{ nearly}$$

$$= \frac{\pi}{2} \cdot D^2 \cdot 2h \text{ nearly (where } h = \text{thickness of shell)}$$

$$= \pi \cdot D^2 \cdot h \text{ nearly}$$

Also, if the thickness of the shell be nearly equal to half the diameter D, that is, if the shell be nearly a solid sphere, $\frac{d}{D}$ is very small, and—

$$V = \frac{\pi}{6} \cdot D^2 \cdot (D - d) \text{ nearly}$$

$$= \frac{\pi}{6} \cdot D^2 \cdot 2h \text{ nearly (where } h = \text{thickness of shell)}$$

$$= \frac{\pi}{3} \cdot D^2 \cdot h \text{ nearly}$$

168. If a sphere be flattened at opposite poles, it is called a *spheroid*.

The Earth is a spheroid.

Such a solid may be seen to be generated by the revolution of an ellipse about one of its axes.

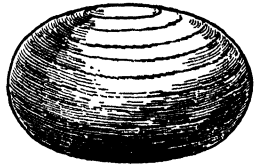


FIG. 1.

FIG. 2.

If the ellipse revolve about its *minor* axis, the spheroid that is generated is said to be *oblate* (see Fig. 1).

If the ellipse revolve about its *major* axis, the spheroid that is generated is said to be *prolate* (see Fig. 2).

A watch may roughly be taken as a familiar example of an oblate spheroid, and a Rugby football of a prolate spheroid.

We have said that spheroids are prismoidal solids (§ 161).

Hence their volumes can be determined by the formula—

$$V = \frac{h}{6}(A_1 + A_2 + 4A) \quad . \quad . \quad . \quad . \quad . \quad \S\ 161.$$

Now, as in the sphere so in the spheroid, we shall regard the points of contact between the spheroid and two parallel tangential planes as two parallel ends of the spheroid.

Hence we may write—

$$A_1 = 0$$
$$A_2 = 0$$

And if $2a$ be the major axis, and $2b$ the minor axis, of the ellipse which, by revolving, generates the spheroid, we may also write—

$$h = 2b, \text{ and } A = \pi a^2$$

when the spheroid is oblate;

$$h = 2a, \text{ and } A = \pi b^2$$

when the spheroid is prolate.

Making these substitutions, we arrive at the following formulæ:—

(1) For the oblate spheroid—

$$V = \frac{2b}{6}(0 + 0 + 4 \cdot \pi a^2)$$
$$= \tfrac{4}{3}\pi a^2 b$$

(2) For the prolate spheroid—

$$V = \frac{2a}{6}(0 + 0 + 4 \cdot \pi b^2)$$
$$= \tfrac{4}{3} \cdot \pi a b^2$$

ILLUSTRATIVE EXAMPLES.

169. *Example* 1.—A sphere $3\frac{1}{2}$ in. in diameter weighs 9 ozs.: find the weight of a cubic foot of the material of which it is made.

Volume of sphere $= \frac{\pi d^3}{6}$ cub. in. § 166.

where $d = 3\frac{1}{2}$;

$\therefore$ volume of sphere $= \frac{\pi \cdot (3\frac{1}{2})^3}{6}$ cub. in.

$= \frac{539}{24}$ cub. in.

$\therefore \frac{539}{24}$ cub. in. of the material weighs 9 ozs.

$\therefore$ 1 cub. ft. of the material weighs

$$\frac{9 \times 1728 \times 24}{539} \text{ ozs.} = 43{\cdot}2801 \text{ lbs.}$$

Example 2.—How many leaden balls of a quarter of an inch in diameter can be cast out of the metal of a ball 3 in. in diameter, supposing no waste?

Volume of each $\frac{1}{4}$-in. ball $= \frac{\pi \times (\frac{1}{4})^3}{6}$ cub. in. . . § 166.

volume of 3-in. ball $= \frac{\pi \times (3)^3}{6}$ cub. in. . . . § 166.

$\therefore$ required number of balls $= \frac{\pi \times (3)^3}{6} \div \frac{\pi \times (\frac{1}{4})^3}{6}$

$= 1728$

Example 3.—A spherical cannon ball, 6 in. in diameter, is melted and cast into a conical mould, the base of which is 12 in. in diameter: find the height of the cone.

Volume of cannon ball $= \frac{\pi d^3}{6}$ cub. in. . . . § 166.

where $d = 6$;

$\therefore$ volume of cannon ball $= \frac{\pi 6^3}{6}$ cub. in.

and if h in. = height of cone

volume of cone $= \frac{1}{3} \cdot \pi \cdot 6^2 \cdot h$ cub. in. . § 142.

$\therefore \frac{1}{3} \cdot \pi \cdot 6^2 \cdot h = \frac{\pi \cdot 6^3}{6}$

or $h = 3$

hence height of cone = 3 in.

Example 4.—If an iron ball of 4 in. diameter weigh 9 lbs., what is the weight of an iron shell 9 in. and 6 in. external and internal diameters respectively?

Solidity of shell $= \frac{\pi}{6}(D^3 - d^3)$ cub. in. § 167

where $D = 9$,
$d = 6$;

$\therefore$ solidity of shell $= \frac{\pi}{6}(9^3 - 6^3)$ cub. in.

Also solidity of ball $= \frac{\pi \cdot 4^3}{6}$ cub. in. § 166.

But the ball weighs 9 lbs.

$$\therefore \text{ the shell weighs } \frac{9 \times \frac{\pi}{6}(9^3 - 6^3)}{\frac{\pi \cdot 4^3}{6}} \text{ lbs.} = 72\tfrac{9}{64} \text{ lbs.}$$

Example 5.—Find the radius of a sphere whose circumference and solid content have the same numerical value.

Let r linear units = radius of sphere.

Then solid content of sphere $= \frac{\pi d^3}{6}$ solid units . . § 166.

where $d = 2r$;

$\therefore$ solid content of sphere $= \frac{4}{3} \cdot \pi r^3$ solid units

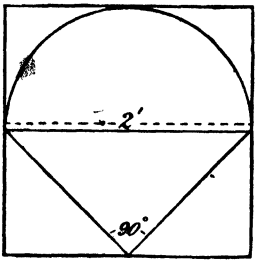
also circumference of sphere $= 2\pi r$ linear units . . . § 69.

hence $\frac{4}{3}\pi r^3 = 2\pi r$

$r^2 = 1{\cdot}5$

$r = 1{\cdot}2247 \ldots$

Example 6.—A right cone and a hemisphere lie on opposite sides of a common base 2 ft. in diameter ; the cone is right-angled at the vertex : if a cylinder circumscribe them in this position, what additional space will be enclosed ?



Consider a vertical mid-section of the three combined figures.

Each figure may be seen to stand upon a base 2 ft. in diameter. The height of the cylinder may be seen to be equal to twice the radius of the hemisphere, and the height of the cone to the radius of the hemisphere.

$\therefore$ volume of hemisphere $= \frac{1}{2} \cdot \frac{\pi \cdot 2^3}{6}$ cub. ft. . . § 166.

volume of cone $= \frac{1}{3} \cdot \pi \cdot 1^2 \cdot 1$ cub. ft. . § 142.

volume of cylinder $= \pi \cdot 1^2 \cdot 2$ cub. ft. . . § 131.

hence additional space enclosed by cylinder $= \{2\pi - (\frac{1}{3}\pi + \frac{2}{3}\pi)\}$ cub. ft.

$= \pi$ cub. ft.

Example 7.—The diameter of a sphere measures 6000 miles : by how many cubic miles approximately does its volume exceed that of another sphere whose diameter is less by 50 yds. ?

Excess of larger sphere over smaller $= \pi \cdot D^2 \cdot h$ cub. mi. approximately § 167.

where $D = 6000$,

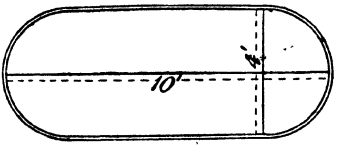
and $h = \frac{25}{1760}$;

$$\therefore \text{ excess} = \frac{\pi \times 6000 \times 6000 \times 25}{1760} \text{ cub. mi. approximately}$$

$$= \frac{22 \times 6000 \times 6000 \times 25}{7 \times 1760} \text{ cub. mi. approximately}$$

$= 1{,}607{,}142\frac{8}{7}$ cub. mi. approximately

Example 8.—A wrought-iron cylindrical boiler 10 ft. long, 4 ft. in diameter, and $\frac{3}{8}$ in. thick, is closed by hemispherical ends: find the weight of metal in it if 1 cub. ft. of wrought iron weigh 496 lbs.



Solidity of boiler = space bounded by outer surface − space bounded by inner surface.

Space bounded by outer surface $= \left\{\frac{\pi(48\frac{3}{4})^3}{6} + \pi(24\frac{3}{8})^2 \times 72\right\}$ cub. in. . §§ 166, 131

$= \pi(24\frac{3}{8})^2\left\{\frac{24\frac{3}{8} \times 8}{6} + 72\right\}$ cub. in.

$= \pi(\frac{195}{8})^2 \times \frac{627}{8}$ cub. in.

Space bounded by inner surface $= \left\{\frac{\pi(48)^3}{6} + \pi(24)^2 . 72\right\}$ cub. in. . §§ 166, 131.

$= \pi(24)^2\left\{\frac{24 \times 8}{6} + 72\right\}$ cub. in.

$= \pi(24)^2 \times \frac{624}{6}$ cub. in.

$= \pi(24)^2 \times 104$ cub. in.

$\therefore$ solidity of boiler $= \pi\{(\frac{195}{8})^2 \times \frac{627}{8} - (24)^2 \times 104\}$ cub. in.

$= \pi \times 2181{\cdot}695$ cub. in.

$= 3{\cdot}9680$ cub. ft.

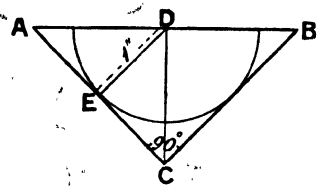
Hence—

weight of metal $= 3{\cdot}9680 \times 496$ lbs.

$= 1968$ lbs. nearly

Note.—The solidity of the boiler can be more easily found, approximately, by multiplying the surface of the boiler by the thickness of the metal.

Example 9.—A conical wineglass whose vertical angle is a right angle, is filled with water. A hemisphere of radius 1 in. is immersed in the water with its convexity downwards, and it is found that when it rests on the sides of the wineglass its flat surface is flush with the level of the water. Find the amount of water left in the glass after the immersion of the hemisphere.



Let the figure represent a vertical mid-section of the cone and hemisphere.

Because the triangles ADC and DEC are isosceles right-angled triangles, and $DE = 1$ in.—

$\therefore DA = DC = \sqrt{2}$ in. . . . . § 17.

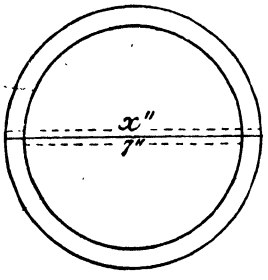
Amount of water left after immersion = cubical content of cone − cubical content of hemisphere

$= \left\{\frac{1}{3}\pi(\sqrt{2})^2 . \sqrt{2} - \frac{\pi(2)^3}{12}\right\}$ cub. in. §§ 142, 166.

$= \frac{2\pi}{3}(\sqrt{2} - 1)$ cub. in.

$= 0{\cdot}8678$ cub. in.

Example 10.—Find the thickness of a shell whose inner diameter measures 7 in., if it weigh half as much as a solid ball of the same diameter.



Let the outer diameter of the shell measure x inches.

Then—

Solidity of shell $= \frac{\pi}{6}(x^3 - 7^3)$ cub. in. § 167.

But solidity of solid ball $= \frac{\pi}{6} \times 7^3$ cub. in. . § 166.

$$\therefore \frac{\pi}{6}(x^3 - 7^3) = \tfrac{1}{2} \times \frac{\pi}{6} \times 7^3$$

$$x^3 = \tfrac{3}{2} \times 7^3$$

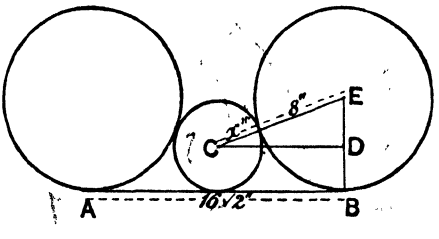
$$x^3 = 514{\cdot}5$$

$$x = 8{\cdot}012$$

hence thickness of shell $= \frac{8{\cdot}012 - 7}{2}$ in.

$$= 0{\cdot}506 \text{ in.}$$

Example 11.—Four balls, each 16 in. in diameter, are placed so that each will touch two of the others: what must be the diameter of a fifth ball, placed on the table in the middle of the four, and touching each of them?



The centres of the four balls when placed in position will be the four corners of a square whose side measures 16 in.

Hence, if the figure represent a vertical section of the balls through a diagonal of this square, and if the radius of the fifth ball measure x in.—

$AB = 16\sqrt{2}$ in. . . . . § 17.

$CD = 8\sqrt{2}$ in.

$CE = (x + 8)$ in.

$DE = (8 - x)$ in.

But $CE^2 = CD^2 + DE^2$ . . . . Euc. I. 47.

$$\therefore (x + 8)^2 = 128 + (8 - x)^2$$

$$32x = 128$$

$$x = 4$$

hence diameter of fifth ball = 8 in.

Examples—XXVII.

(Take $\pi = \frac{22}{7}$, unless otherwise stated.)

Find the volumes of spheres having the following diameters :—

1. 9 ft.

2. 3 ft. 2 in.

3. 6·125 in.

4. 1 yd. 2 ft. 7 in.

Find the diameters of spheres having the following volumes :—

5. $179\frac{3}{4}$ cub. ft.

6. $113\frac{1}{7}$ cub. in.

Find in cubic inches to two places of decimals the volumes of spheres having the following circumferences :—

7. 10 in.

8. 2 ft. 4 in.

9. How many spherical bullets can be made out of a cube of lead whose edge measures 10 in., each bullet being 1 in. in diameter?

10. How many gallons of water will a hemispherical bowl contain whose radius is 2 ft.?

11. Find the weight of a solid metal sphere of radius 8 in., if 1 cub. in. of the metal weigh 8 ozs. (Avoir.).

12. A hemispherical tank is emptied by a pipe at the rate of 8 galls. per minute: how long will it take to half empty the tank if it is 1 ft. 6 in. in diameter?

13. Find the volume of a solid in the form of a right circular cylinder with hemispherical ends whose extreme length is 24 ft. and diameter 2 ft. 6 in.

14. What weight of powder will be required to fill a spherical shell whose internal diameter is 18 in., if 30 cub. in. of powder weigh 1 lb.?

15. Find the volume of a spherical shell whose internal and external diameters are 8 in. and 10 in. respectively.

16. The internal diameter of a spherical shell is 10 in., and its thickness $\frac{1}{2}$ in.: find the weight of the shell if it is composed of a substance weighing 180 lbs. per cubic foot.

17. The external diameter of a spherical shell is 10 in., and its thickness 1 in.: find its weight if it is made of copper weighing 5 ozs. per cubic inch.

18. An ellipse, whose major axis measures 2 ft. 8 in., and whose minor axis measures 1 ft. 6 in., revolves about its minor axis: find the volume of the spheroid generated.

19. An ellipse, whose major axis measures 7·42 in., and whose minor axis measures 3·82 in., revolves about its major axis: find the volume of the spheroid generated. ($\pi = 3{\cdot}1416$.)

Examination Questions—XXVII.

(Take $\pi = \frac{22}{7}$, unless otherwise stated.)

A. *Allahabad University: Intermediate.*

1. A solid consisting of a right cone standing on a hemisphere is placed in a right cylinder full of water, and touches the bottom: find the volume of water displaced, having given that the radius of the cylinder is 3 ft. and its height 4 ft., the radius of the hemisphere 2 ft., and the height of the cone 4 ft.

2. Assuming a drop of water to be spherical and $\frac{1}{10}$ in. in diameter, to what depth will 1000 drops fill a conical wineglass, the cone of which has a height equal to the diameter of its rim?

B. *Bombay University: L.C.E. Second Exam.*

3. What three integers are proportional to the volume of a sphere, that of its circumscribing cylinder, and that of its circumscribing equilateral cone?

4. Deduce the volume of a sphere from that of a cone.

C. *Punjab University: First Exam. in Civil Engineering.*

5. A sphere is 36 in. in diameter: find its volume in cubic feet.

6. A spherical cannon ball, 9 in. in diameter, is melted and cast into a conical mould, the base of which is 18 in. in diameter: find the height of the cone.

7. What is the content of a sphere whose diameter is 21 in.?

D. *Madras University: B.E. Exam.*

8. It is calculated that the heat received by the earth from the sun in a year would be sufficient to melt a layer of ice 100 ft. thick all over the surface of the earth: assuming the earth to be a sphere of radius 4000 miles, find the volume of this ice in cubic miles.

E. *Calcutta University: F.E. Exam.*

9. A hollow shell 12 in. in external diameter is placed in a conical vessel whose vertical angle is 60°, and water poured into it until it just covers the shell and fills the cavity in it. When the shell emptied of water is removed and a solid ball of the same diameter substituted for it, the water stands $\frac{1}{2}$ in. above it. Find approximately the thickness of the shell.

10. A solid iron cube, the edge of which is 2 ft. in length, and a solid iron sphere the radius of which is 1 ft., are thrown into a cubical tank which is 6 ft. across, and is half filled with water: find the rise of the surface of the water in inches to five places of decimals if they both be completely immersed. ($\pi = 3{\cdot}14159$.)

11. From a cylinder whose height is equal to its diameter the greatest possible sphere is turned: find what fraction of the original solid has been cut away.

F. *Sibpur Apprentice Dept.: Monthly Exam.*

12. Find the weight of a pyramid of iron such that its height is 8 in., and its base is an equilateral triangle, each side being 2 in., supposing a ball of iron 4 in. in diameter to weigh 9 lbs.

13. A hemispherical basin 15 ft. in diameter will hold one hundred and twenty times as much as a cylindrical tub, the depth of which is 1 ft. 6 in.: find the diameter of the tub.

14 A solid ball 4 in. in radius of a certain material weighs 8 lbs.: find the weight of a spherical shell of that material, the internal diameter of which is 8 in. and the external diameter 10 in.

15. Find the radius of the base of a cone which has the same volume as a sphere of 5 ft. radius, and the height of the cone one-half of the radius of the sphere.

16. If the diameter of the earth be 8000 miles, and geologists knew the interior to the depth of 5 miles below the surface, what fraction of the whole contents would be known?

17. How many spherical bullets, each one $\frac{1}{2}$ in. in diameter, can be cast from a rectangular block of lead 1 ft. 3 in. by 1 ft. 2 in. by 5 in.?

18. The radius of the base of a cone is 4 in.: find the height so that the volume may be equal to that of a sphere with diameter 4 in.

G. *Sibpur Apprentice Dept.: Annual Exam.*

19. A solid is composed of a cone and hemisphere on opposite sides of the circular base, the diameter of which is 2 ft., and the vertical angle of the cone is a right angle: find the volume of the whole.

20. A hemispherical punch-bowl is 5 ft. 6 in. round the brim: supposing it to be half full, how many persons may be served from it in hemispherical glasses $1\frac{3}{4}$ in. in diameter at the top?

H. *Sibpur Apprentice Dept.: Final.*

21. A 2-ft. tube is partly filled with water, a sphere that exactly fits the tube is placed in it, and the water is found to rise just to the highest point of the sphere: how much water was there in the tube?

22. A cast-iron shell has an external diameter of 1 ft. The metal is 2 in. thick. Find the weight of the shell. (1 cub. ft. of iron weighs 450 lbs.)

I. *Roorkee Engineer: Entrance.*

23. A hemisphere of lead of radius 6 in. is cast into a solid cube: find to three decimal places the length of an edge of the cube. ($\pi = 3{\cdot}14159$.)

24. Find the thickness of a shell whose outer diameter measures 7 in., if it weigh half as much as a solid ball of the same diameter.

25. Find the weight of a hollow iron shell, if the exterior diameter is 13 in. and the thickness of the iron 2 in. (Iron weighs 4·2 ozs. per cubic inch.)

26. If a spherical shell when formed into a solid sphere be equal in volume to its own cavity, what must be the thickness of the shell?

J. *Roorkee Upper Subordinate: Entrance.*

27. A hemisphere and a right cone lie on opposite sides of a common base of 4 ft. diameter, and the cone is right-angled at the vertex: if a cylinder should circumscribe them in this position, how much additional space is thereby enclosed?

28. If 30 cub. in. of gunpowder weigh 1 lb., find the diameter of a hollow sphere which will hold 11 lbs.

29. Find how many gallons of water a hemispherical bowl 2 ft. 4 in. in diameter will hold.

30. A heavy iron cylinder with hemispherical ends is immersed in water: find the amount of water displaced, the solid's extreme length being 12 ft. and diameter 3 ft.

31. The external and internal diameters of a shell are respectively $15\frac{1}{2}$ in. and $10\frac{3}{4}$ in.: find the volume.

32. Find the weight of a 13-in. iron shell, the thickness of which is 2 in., the weight of a cubic foot of iron being 441 lbs.

K. *Roorkee Engineer: Final.*

33. What is the weight of an iron shell, the external and internal diameters of which are 13 and 10 in. respectively, if an iron ball of 4 in. diameter weigh 9 lbs.?

34. A hemispherical bowl whose internal radius is 1 ft. is filled with water and kept so that the rim is horizontal. A cone whose vertical angle is 90° is placed with its axis vertical, its base at the level of the rim of the bowl, and its apex at the centre of the bottom of the bowl. Find the quantity of water left in the bowl after the intrusion of the cone.

L. *Staff College.*

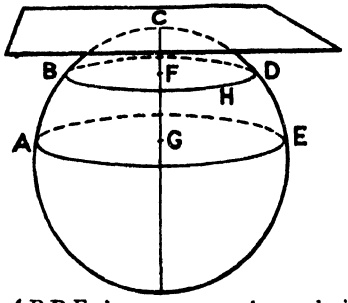
35. Determine the weight of a spherical cast-iron shell whose inner and outer diameters are $6\frac{1}{2}$ and $7\frac{1}{2}$ in. respectively, the weight of a cubic foot of iron being 450 lbs.

CHAPTER XXVIII.

ON ZONES OF SPHERES, SEGMENTS OF SPHERES, AND SECTORS OF SPHERES.

170. A *zone of a sphere* is the part of a sphere included between two parallel planes.

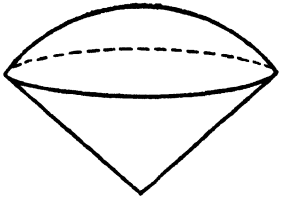
The *height* of a zone is the perpendicular distance between the two parallel planes. When one of the parallel planes does not *cut* the sphere, but only *touches* it, the zone is called a *segment* of the sphere.



Hence a segment of a sphere may be defined as the figure bounded by a plane and the part of the sphere it cuts off.

The *base* of the segment is the section of the sphere by the plane.

Thus, in the sphere *ABCDE*, *ABDE* is a zone whose height is *FG*, *BCD* is a segment whose height is *CF* and whose base is *BHD*.



171. A *sector of a sphere* is the solid composed of a segment of the sphere and the cone having the base of the segment for its base and the centre of the sphere for its vertex.

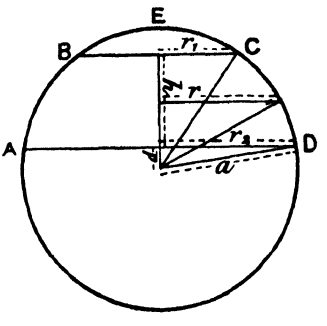
PROPOSITION XL.

172. *To find the volume of a zone of a sphere, having given the radii of the two ends and the height of the zone.*

Let *ABCD* be a zone of the sphere *ABECD*.

Let the radii of its two ends and the height of the zone measure r_1, r_2, and h of the same linear unit respectively.

It is required to find the volume of the zone in terms of r_1, r_2, and h. Let the radius of the sphere measure a, and the radius of the mid-section of the zone measure r, and the perpendicular from the centre of the sphere on to the nearest end of the zone measure p of the same linear unit.



Since a zone of a sphere is a prismoidal solid (§ 161), therefore the formula—

$$V = \frac{h}{6}(A_1 + A_2 + 4A)$$

will determine its volume . . . . . § 161.

But here we have—

$$\left.\begin{aligned} A_1 &= \pi r_1^2 \\ A_2 &= \pi r_2^2 \\ A &= \pi r^2 \end{aligned}\right\} \quad \ldots\ldots\ldots \S\ 71$$

$$\therefore \text{ volume of zone } ABCD = \frac{h}{6}(\pi r_1^2 + \pi r_2^2 + 4\pi r^2) \text{ solid units}$$

If, then, we can express r in terms of r_1, r_2, and h, what was required will have been done.

Now—

$$\left.\begin{aligned} r^2 + \left(p + \frac{h}{2}\right)^2 &= a^2 \\ r_2^2 + p^2 &= a^2 \\ r_1^2 + (p + h)^2 &= a^2 \end{aligned}\right\} \quad \ldots\ldots\ldots \S\ 16.$$

$$\left.\begin{aligned} \therefore r^2 + p^2 + ph + \frac{h^2}{4} &= a^2 \\ r_2^2 + p^2 &= a^2 \\ r_1^2 + p^2 + 2ph + h^2 &= a^2 \end{aligned}\right\}$$

Therefore, subtracting—

$$\left.\begin{aligned} r^2 + ph + \frac{h^2}{4} - r_2^2 &= 0 \\ r_1^2 + 2ph + h^2 - r_2^2 &= 0 \end{aligned}\right\}$$

Therefore eliminating p—

$$r^2 = \frac{h^2}{4} + \frac{r_1^2 + r_2^2}{2}$$

Hence, substituting for r^2, we have—

$$\text{Volume of zone } ABCD \Big\} = \frac{\pi h}{6}\left\{r_1^2 + r_2^2 + 4\left(\frac{h^2}{4} + \frac{r_1^2 + r_2^2}{2}\right)\right\} \text{ solid units}$$

$$= \frac{\pi h}{6}\{3(r_1^2 + r_2^2) + h^2\} \text{ solid units}$$

Hence rule—

To three times the sum of the squares of the numbers of any linear unit in the radii of the ends of a zone of a sphere, add the square of the number of the same linear unit in the height, multiply this sum by the number of the same linear unit in the height, then the product multiplied by $\frac{\pi}{6}$ *will give the number of the corresponding solid unit in the volume.*

Or briefly—

Vol. of zone of sphere $= \frac{\pi}{6} \times$ **height** $\times$ **(3 $\times$ sum of squares of radii of ends + height^2).**

$$\mathbf{V = \frac{\pi h}{6}\{3(r_1^2 + r_2^2) + h^2\}}$$

PARTICULAR CASE.

173. Segment of sphere.

Here $r_2 = 0$ . . . . . § 170.

$\therefore$ vol. of segment of sphere $= \frac{\pi h}{6}(3r_1^2 + h^2)$ solid units

Again, if the diameter of the sphere measure d linear units, we have—

$$r_1^2 = h(d - h) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{§ 75.}$$

And the above formula may be written—

$$V = \frac{\pi h}{6}\{3h(d - h) + h^2\}$$

$$\text{or } V = \frac{\pi h^2}{6}(3d - 2h)$$

If the height of the segment be equal to the radius of the sphere, the segment is a hemisphere. Write, then, $h = \frac{d}{2}$ in the formula—

$$\frac{\pi h^2}{6}(3d - 2h)$$

and we have—

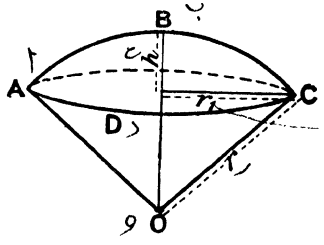
$$\text{Volume of hemisphere} = \frac{\pi}{6} \cdot \left(\frac{d}{2}\right)^2 \cdot (3d - d) \text{ solid units}$$

$$= \frac{\pi d^3}{12} \text{ solid units}$$

This agrees with the result previously obtained in § 166.

PROPOSITION XLI.

174. *To find the volume of a sector of a sphere, having given the height of its segment and the radius of the sphere.*



Let $OABCD$ be a sector of a sphere.

Let the height of its segment $ABCD$ and the radius of the sphere measure h and r of the same linear unit respectively.

It is required to find the volume of the sector in terms of h and r.

Let the radius of the base of the segment measure r_1 of the same linear unit.

$$\left.\begin{matrix}\text{Now, volume of} \\ \text{sector } OABCD\end{matrix}\right\} = \text{vol. of segment } ABCD + \text{vol. of cone } OACD$$

$$= \frac{\pi h}{6}(3r_1^2 + h^2) \text{ solid units} + \tfrac{1}{3}(r - h)\ \pi r_1^2 \text{ solid units} \quad \S\S\ 173, 142.$$

$$\text{But } r_1^2 = h(2r - h) \quad \S\ 75.$$

$$\left.\begin{matrix}\therefore \text{ volume of} \\ \text{sector}\end{matrix}\right\} = \frac{\pi h}{6}\{3h(2r - h) + h^2\} \text{ solid units} + \tfrac{1}{3}(r - h) \cdot \pi \cdot h(2r - h) \text{ solid units}$$

$$= \frac{\pi h}{6}(6hr - 2h^2) \text{ solid units} + \frac{\pi h}{3}(2r^2 - 3hr + h^2) \text{ solid units}$$

$$= \frac{\pi h}{3}(3hr - h^2 + 2r^2 - 3hr + h^2) \text{ solid units}$$

$$= \tfrac{2}{3}\pi r^2 h \text{ solid units}$$

Hence rule—

The product of the number of any linear unit in the height of the

segment of a sector of a sphere by the square of the number of the same linear unit in the radius of the sphere, multiplied by $\frac{2}{3}\pi$, will give the number of the corresponding solid unit in the volume of the sector.

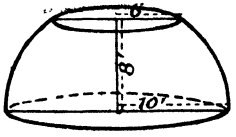
Or briefly—

$$\text{Vol. of sector of sphere} = \begin{cases} \frac{2}{3}\pi \times (\text{radius of sphere})^2 \\ \quad \times (\text{height of segment}) \end{cases}$$

$$V = \tfrac{2}{3}\pi r^2 h$$

ILLUSTRATIVE EXAMPLES.

175. *Example* 1.—In a spherical zone the radii of the two ends are 10 ft. and 6 ft., the altitude 8 ft.: find the solid content.

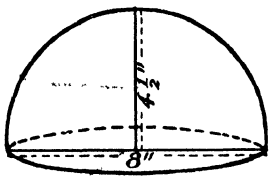


Volume of zone $= \frac{\pi h}{6}\{3(r_1^2 + r_2^2) + h^2\}$ cub. ft. . . . . § 172.

where $h = 8$,
$r_1 = 10$,
$r_2 = 6$;

$\therefore$ volume of zone $= \frac{22}{7} \times \frac{8}{6}\{3(100 + 36) + 64\}$ cub. ft. nearly
$= \frac{88}{21} \times 472$ cub. ft. nearly
$= 1977{\cdot}9$ cub. ft. nearly

Example 2.—Find the volume of a segment of a sphere whose height is $4\frac{1}{2}$ in., and the diameter of whose base is 8 in.



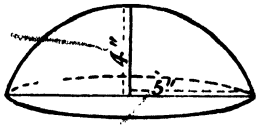
Volume of segment $= \frac{\pi h}{6}(3r_1^2 + h^2)$ cub. in. . . § 173.

where $h = 4\frac{1}{2}$,
$r_1 = 4$;

$\therefore$ volume of segment $= \frac{\pi \times 9}{2 \times 6}\{3 \times 4^2 + (\frac{9}{2})^2\}$ cub. in.

$= \frac{22 \times 9 \times 273}{7 \times 2 \times 6 \times 4}$ cub. in. nearly
$= 160{\cdot}8$ cub. in. nearly

Example 3.—The radius of the base of a segment of a sphere is 5 in. and the height 4 in.; if the segment is beaten out into a circular plate 40 in. in diameter, what is the thickness of the plate?



Volume of segment $= \frac{\pi h}{6}(3r_1^2 + h^2)$ cub. in. § 173.

where $r_1 = 5$,
$h = 4$;

$\therefore$ volume of segment $= \frac{\pi \times 4}{6}(3 \times 5^2 + 4^2)$ cub. in.

$= \frac{182\pi}{3}$ cub. in.

hence, if x in. = thickness of plate

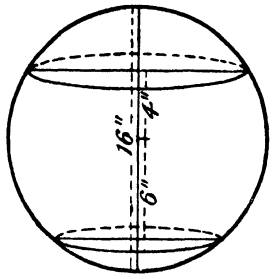
$$\pi(20)^2 . x = \frac{182\pi}{3} \quad \ldots \ldots \ldots \S\ 131.$$

$$x = \frac{91}{600}$$

$$= 0{\cdot}1516$$

that is, the thickness of the plate = 0·1516 of an inch.

Example 4.—What proportion of the volume of a sphere 16 in. in diameter is contained between two parallel planes distant 4 and 6 in. from the centre and on opposite sides of it?



The portion of the sphere contained between the two parallel planes is a zone of the sphere.

Let r_1 in. and r_2 in. be the radii of the ends of this zone.

Then—

$$r_1^2 = 4 \times 12 = 48 \ldots \S\ 75.$$

$$r_2^2 = 2 \times 14 = 28 \ldots \S\ 75.$$

$$\therefore \text{ the volume of this zone} = \frac{\pi h}{6}\{3(r_1^2 + r_2^2) + h^2\} \text{ cub. in.} \quad \ldots \S\ 172.$$

where $h = 10$,

$r_1^2 = 48$,

$r_2^2 = 28$;

$$\text{hence volume of zone} = \frac{\pi \times 10}{6}\{3(48 + 28) + 100\} \text{ cub. in.}$$

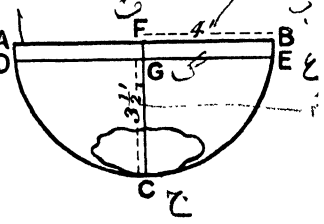
$$= \frac{5\pi}{3} \times 328 \text{ cub. in.}$$

$$\text{and volume of whole sphere} = \frac{\pi \times 16^3}{6} \text{ cub. in.}$$

$$\therefore \text{ required proportion} = \frac{\frac{5\pi}{3} \times 328}{\frac{\pi \times 16^3}{6}}$$

$$= \frac{205}{256}$$

Example 5.—A stone was rolled into a hemispherical basin 8 ft. diameter having $3\frac{1}{2}$ ft. depth of water in it, when the water immediately rose to the lip of the basin. What was the cubic content of the stone?



Let the figure represent a vertical mid-section of the basin and stone.

Let DE and AB indicate the level of the water before and after the immersion of the stone.

Then—

$GC = 3\frac{1}{2}$ ft.

$FG = \frac{1}{2}$ ft.

$FB = 4$ ft.

And if $GE = x$ ft.—

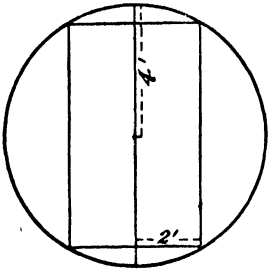
$$x^2 = 3\tfrac{1}{2} \times 4\tfrac{1}{2} \quad \ldots\ldots\ldots \text{§ 75.}$$
$$= \frac{63}{4}$$
$$x = \frac{\sqrt{63}}{2}$$
$$\therefore GE = \frac{\sqrt{63}}{2} \text{ ft.}$$

Now, the cubical content of the stone = the cubical content of the water displaced

= the cubical content of zone $ABED$

$$= \frac{\pi(\frac{1}{2})}{6}\{3(\tfrac{63}{4} + 16) + (\tfrac{1}{2})^2\} \text{ cub. ft.} \quad \ldots\ldots \text{§ 172.}$$
$$= \frac{\pi}{12} \times 18\tfrac{1}{2} \text{ cub. ft.}$$
$$= 25{\cdot}011 \text{ cub. ft. nearly}$$

Example 6.—A sphere of 4 ft. radius has to have a cylinder of 2 ft. radius put through it centrically. Find the volume of the sphere that has to be cut out ($\pi = 3{\cdot}1416$).



Let the figure represent a vertical mid-section of the sphere and cylinder.

Volume of sphere cut out = volume of two equal segments of the sphere + volume of a cylinder.

Let x ft. = height of each segment.

Then—

$$x(8 - x) = 4 \quad \ldots \text{§ 75.}$$
$$x = 4 - 2\sqrt{3}$$
$$\therefore \text{height of each segment} = (4 - 2\sqrt{3}) \text{ ft.}$$
$$\text{and length of cylinder} = \{8 - 2(4 - 2\sqrt{3})\} \text{ ft.}$$
$$= 4\sqrt{3} \text{ ft.}$$

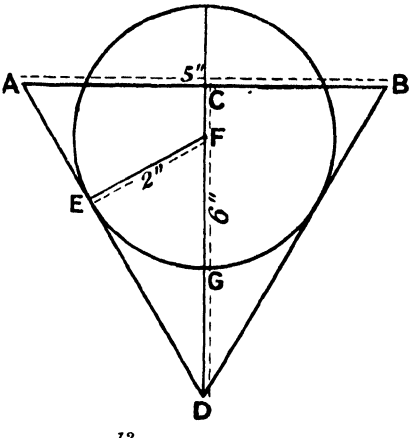
Hence—

$$\text{Vol. of each segment} = \frac{\pi(4 - 2\sqrt{3})}{6}\{3 \times 2^2 + (4 - 2\sqrt{3})^2\} \text{ cub. ft.} \quad \text{§ 173.}$$
$$= \frac{\pi(2 - \sqrt{3})}{3} \cdot (40 - 16\sqrt{3}) \text{ cub. ft.}$$

and—

$$\text{volume of cylinder} = \pi \cdot 2^2 \cdot 4\sqrt{3} \text{ cub. ft.} \quad \ldots\ldots\ldots \text{§ 131.}$$
$$= 16\sqrt{3} \cdot \pi \text{ cub. ft.}$$
$$\therefore \text{volume of sphere cut out} = \left\{\frac{2\pi(2 - \sqrt{3})}{3} \times (40 - 16\sqrt{3}) + 16\sqrt{3} \cdot \pi\right\} \text{ cub. ft.}$$
$$= 16\pi\left\{\frac{(2 - \sqrt{3})(5 - 2\sqrt{3})}{3} + \sqrt{3}\right\} \text{ cub. ft.}$$
$$= 16\pi(0{\cdot}13715 + 1{\cdot}73205) \text{ cub. ft.}$$
$$= 16 \times 3{\cdot}1416 \times 1{\cdot}8692 \text{ cub. ft.}$$
$$= 93{\cdot}95 \text{ cub. ft. nearly}$$

Example 7.—A conical glass, whose depth is 6 in., and the diameter of whose mouth is 5 in., being filled with water, and a sphere 4 in. diameter, of greater specific gravity than water, being put into it, how much water will run over?



Let the figure represent a vertical mid-section of the glass and sphere.

By similar triangles—

$DE : EF = CD : AC$ § 66.

$DE : 2 \text{ in.} = 6 : 2\frac{1}{2}$

$\therefore DE = \frac{24}{5} \text{ in.}$

$\therefore FD = \sqrt{(\frac{24}{5})^2 + 2^2} \text{ in.}$ § 16.

$= \frac{26}{5} \text{ in.}$

$\therefore CF = (6 - \frac{26}{5}) \text{ in.} = \frac{4}{5} \text{ in.}$

and $CG = 2\frac{4}{5} \text{ in.}$

Volume of segment immersed $= \frac{\pi h^2}{6}(3d - 2h)$ cub. in. . § 173.

where $h = 2\frac{4}{5}$,

$d = 4$.

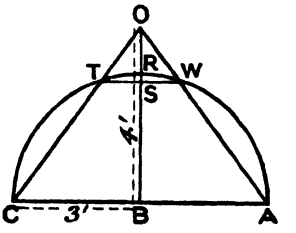
$\therefore$ volume of segment immersed $= \frac{\pi}{6} \times \frac{196}{25}(12 - \frac{28}{5})$ cub. in.

$= \frac{22 \times 196 \times 32}{7 \times 6 \times 25 \times 5}$ cub. in. nearly

$= 26{\cdot}28$ cub. in. nearly

= amount of water that will run over

Example 8.—A hemisphere standing upon its circular base of 3 ft. radius is pierced by a cone 4 ft. high standing on the same base: find the cubical content of the portion of the hemisphere left.



Let the figure represent a vertical mid-section of the hemisphere and cone.

Then—

$OB = 4$ ft.

$BC = BR = 3$ ft.

$OR = 1$ ft.

Let $OS = 4x$ ft. Then—

$TS = 3x$ ft. . . . . § 66.

$RS = (4x - 1)$ ft.

$SB = (4 - 4x)$ ft.

Now, $(4x - 1)(7 - 4x) = 9x^2$ . . . . . § 75.

$\therefore x = \frac{7}{25}$

$\therefore TS = \frac{21}{25}$ ft.

$RS = \frac{3}{25}$ ft.

$SB = \frac{72}{25}$ ft.

Volume of frustum $TWCA$ $= \pi \times \frac{12}{25} \times \frac{1}{3}\left\{(\frac{21}{25})^2 + (3)^2 + \frac{3 \times 21}{25}\right\}$ cub. ft. § 162.

$= \pi \times \frac{12}{25} \times \frac{1}{3} \times \frac{7641}{625}$ cub. ft.

volume of segment $TRWS$ $= \pi \times \frac{3}{25} \times \frac{1}{6}\{3 \times (\frac{21}{25})^2 + (\frac{3}{25})^2\}$ cub. ft. . § 173.

$= \pi \times \frac{3}{25} \times \frac{1}{6} \times \frac{1332}{625}$ cub. ft.

$\therefore$ total volume cut away $= \frac{\pi \times 3 \times 9 \times 2}{25 \times 3 \times 625}(12 \times 849 + 37)$ cub. ft.

$= \frac{7362\pi}{625}$ cub. ft.

volume of whole hemisphere $= \frac{\pi \cdot 6^3}{12}$ cub. ft. . . . . § 166.

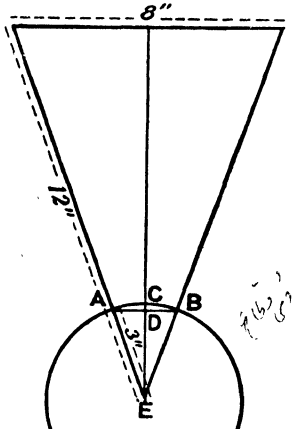
$= 18\pi$ cub. ft.

$\therefore$ volume of part left $= \pi(18 - \frac{7362}{625})$ cub. ft.

$= \frac{22}{7} \times \frac{3888}{625}$ cub. ft. nearly

$= 19{\cdot}55$ cub. ft. nearly

Example 9.—A right cone, base 8 in. diameter, and slant height 12 in., is set into a sphere of radius 3 in., so that vertex and centre coincide: find the volume of the solid.



Let the figure represent a section of the solid through the axis of the cone.

Height of cone $= \sqrt{144 - 16}$ in. § 16.

$= 8\sqrt{2}$ in.

By similar triangles—

$ED : 8\sqrt{2}$ in. $= 3 : 12$. § 66.

$\therefore ED = 2\sqrt{2}$ in.

$\therefore CD = (3 - 2\sqrt{2})$ in.

Hence—

Spherical sector $EACB$ $= \frac{2}{3} \cdot \pi \cdot (3)^2 \cdot (3 - 2\sqrt{2})$ cub. in. . . § 174.

$= 6 \times \pi \times 0{\cdot}17157$ cub. in.

$= \pi \times 1{\cdot}029$ cub. in.

so cone $= \pi \cdot (4)^2 \cdot \frac{8\sqrt{2}}{3}$ cub. in. § 142.

$= \pi \times 60{\cdot}339$ cub. in.

and sphere $= \frac{\pi \cdot (6)^3}{6}$ cub. in. § 166.

$= \pi \times 36$ cub. in.

$\therefore$ required volume $= \pi(36 + 60{\cdot}339 - 1{\cdot}029)$ cub. in.

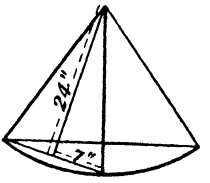
$= 299{\cdot}5$ cub. in. nearly

Example 10.—A cone 24 in. high, the diameter of whose base is 14 in., is set spinning round on an axis, which is a line joining the vertex and any point in the circumference of the base: find the volume of the figure described.

The figure described is a spherical sector. Consider a mid-section of this sector through the centre of the sphere.

The radius of the sphere measures $\sqrt{(24)^2 + (7)^2}$ in. . . . § 16.
$= 25$ in.

and, if the height of the segment of the sector measure h in., and the radius of the base of this segment measure c in., we have—



$$h(50 - h) = c^2 \quad . \quad . \quad . \quad . \quad \S 75.$$
$$\text{and } c^2 = (14)^2 - h^2 \quad . \quad \S 16.$$
$$\therefore h(50 - h) = (14)^2 - h^2$$
$$h = \tfrac{98}{25}$$
$$\therefore \text{volume of spherical sector} = \tfrac{2}{3} \cdot \pi \cdot r^2 h \text{ cub. in.} \quad \S 174.$$

where $r = 25$,
$h = \frac{98}{25}$;
hence volume of spherical sector $= \frac{2}{3} \cdot \frac{22}{7} \cdot (25)^2 \cdot \frac{98}{25}$ cub. in. nearly
$= 5133\frac{1}{3}$ cub. in. nearly

Examples—XXVIII.

(Take $\pi = \frac{22}{7}$.)

1. The radii of the ends of a zone of a sphere are 5 in. and 6 in. respectively; the height is 4 in.: find the volume.

2. The radii of the ends of a zone of a sphere are 9 in. and 10 in. respectively; the height is 6 in.: find the volume.

3. The radius of the base of a segment is 3 in., and the height is 2 in.: find the volume.

4. The radius of the base of a segment is 2 ft. 6 in., and the height 9 in.: find the volume.

5. The height of a segment of a sphere is 3 ft., and the diameter of the sphere is 10 ft.: find the volume of the segment.

6. The height of a segment of a sphere is 6 in., and the radius of the sphere is 2 ft.: find the volume of the segment.

7. A sphere, whose diameter is 18 in., is cut through by a plane, the perpendicular distance of which from the centre is 3 in.: find the volumes of the two segments into which the sphere is divided.

8. A sphere whose diameter is 10 in. is divided into three parts of equal heights by two parallel planes: find the volume of each part.

9. Find the volume of a zone cut from a sphere of diameter 6 in. by two parallel planes both on the same side of the centre and distant from it 1·5 and 2 in. respectively.

10. Find the volume of a zone cut from a sphere of diameter 3 ft. 2 in. by two parallel planes, on opposite sides of the centre, and distant from it 10 in. and 7 in. respectively.

11. Find to the nearest gallon the quantity of water that is contained in a bowl whose shape is a segment of a sphere; the depth of the bowl is 7 in., and the radius of the top 11 in.

12. A sphere, whose diameter is 10 ft., is divided into four parts of equal heights by three parallel planes: find the volume of each part.

13. Find the volume of a spherical sector of the following dimensions: radius of sphere 2 ft. 11 in., height of segment 9 in.

14. Find the volume of a spherical sector of the following dimensions: radius of sphere 14 in., height of segment 2 in.

Examination Questions—XXVIII.

(Take $\pi = \frac{22}{7}$, unless otherwise stated.)

Zones of Spheres.

A. *Bombay University L.C.E.: Second Exam.*

1. A spherical zone is 4 ft. thick, and the diameters of its opposite faces are 12 ft. and 8 ft.: find the volume.

2. What proportion of the volume of a sphere 12 in. in diameter is contained between two parallel planes distant 2 in. and 4 in. from the centre and on the same side of it?

B. *Sibpur Apprentice Dept.: Monthly Exam.*

3. Find the volume of a spherical zone, the radii of the two ends being 3 ft. and 2 ft. respectively, and the height of the zone $1\frac{1}{2}$ ft.

C. *Sibpur Apprentice Dept.: Annual Exam.*

4. A hemispherical bowl of 6 ft. diameter is partially buried with its mouth downwards, and in a horizontal position, so that only one-third of the height appears above the ground: find what quantity of earth must be dug out in order to leave the bowl entirely uncovered and just surrounded by a cylindrical wall of earth.

D. *Roorkee Engineer: Entrance.*

5. Find the volume of a zone of a sphere, supposing the ends to be on the same side of the centre of the sphere, and distant respectively 10 in. and 15 in. from the centre, and the radius of the sphere to be 20 in.

E. *Roorkee Upper Subordinate: Monthly.*

6. A sphere, 16 in. in diameter, is divided into four parts of equal heights by three parallel planes: find the volume of each part. ($\pi = 3{\cdot}1416$.)

7. A globe 18 in. in diameter is divided into three portions of equal heights by parallel planes: what are the volumes of these portions?

F. *Roorkee Engineer: Final.*

8. What is the solidity of a zone whose greater diameter is 9 ft. 3 in., lesser diameter 6 ft. 9 in., and height 5 ft. 6 in.?

Segments of Spheres.

A. *Bombay University L.C.E.: Second Exam.*

9. Find the weight of a dumb-bell consisting of segments of two spheres of 5 in. diameter joined by a cylindrical bar 7 in. long and 2 in. in diameter, an iron ball 4 in. in diameter weighing 9 lbs.

10. A Stilton cheese is cylindrical in shape, while a Dutch cheese is spherical: find the height of a Stilton cheese which stands on a base 6 in. in diameter, and is equal in solid content to a segment of a Dutch cheese $4\frac{1}{2}$ in. thick, the original diameter of the Dutch cheese being $13\frac{3}{4}$ in.

B. *Madras University: B.E. Exam.*

11. Find the volume of a segment of a sphere when the radius of the base is 16 ft., and height of segment 5 ft.

C. *Sibpur Apprentice Dept.: Monthly Exam.*

12. The diameter of a sphere is 18 ft.; the sphere is divided into two segments, one of which is twice as high as the other: find the volume of each.

13. A bowl is in the shape of a segment of a sphere; the depth of the bowl is 9 in., and diameter of the top of the bowl is 3 ft.: find to the nearest gallon the quantity of water the bowl will hold.

D. *Sibpur Apprentice Dept.: Annual Exam.*

14. If a heavy sphere whose diameter is 4 in. be put into a conical glass full of water whose diameter at the rim is 5 in. and depth 6 in., how much of the water will run over?

E. *Sibpur Apprentice Dept.: Final Exam.*

15. The height of a segment of a sphere is 5 ft. and the diameter of sphere is 15 ft.: find the volume.

F. *Roorkee Upper Subordinate: Entrance.*

16. The radius of the base of a segment of a sphere is 1 in., and the radius of the sphere is $2\frac{1}{2}$ in.: find the volume of the segment.

17. The height of a segment of a sphere is 2 ft. 3 in., and the diameter of the sphere is 6 ft. 3 in.: find the volume.

18. Find the edge of the greatest cube that can be cut out of the segment of a sphere the radius of whose base is 10 in. and the height 5 in.

Sectors of Spheres.

A. *Bombay University L.C.E.: First Exam.*

19. A solid sector is cut out of a sphere of 10 ft. radius by a cone, the angle of which is 120°: find the radius of the sphere whose solid contents are equal to those of the sector.

B. *Roorkee Engineer: Entrance.*

20. A cone is inserted in a sphere, the vertex of the cone being in the centre of the sphere. The diameter of the sphere is 24 in. The radius of base of cone is 15 in., and height 20 in. What weight does the sphere lose, 1 cub. in. of the material weighing 0·25 lb.?

C. *Roorkee Engineer: Final.*

21. A cone 4 ft. high, the diameter of whose base is 6 ft., is set spinning round on an axis which is a line joining the vertex and any point in the circumference of the base: find the volume of the figure described.

Additional Examination Questions—XXVIII.

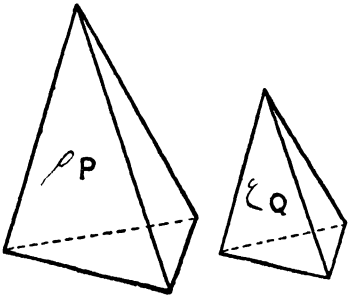
22. Two tangents, OP, OQ, are drawn to a circle of radius 8 in. at P and Q, the angle between them being 60°. If the figure is made to revolve about the line joining O to the centre of the circle, find the volume of the solid so formed. (Roorkee Upper Subordinate: Entrance.)

23. A zone of a sphere, diameter 2 ft., has for its ends two parallel circles distant respectively 3 in. and 9 in. from a common pole: find the volume of this zone. (Roorkee Engineer: Entrance.)

CHAPTER XXIX.

ON SIMILAR SOLIDS.

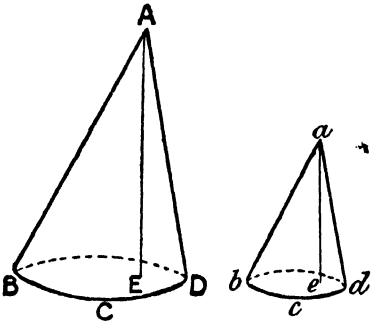
176. Solids are said to be *similar* when they are of the same shape, though they need not be of the same size. Thus the pyramid P is similar to the pyramid Q.



All cubes are similar to one another, and so are all spheres.

A model of an engine is similar to the engine itself. Any solid is similar to itself when magnified.

If a smaller pyramid be cut off from a larger pyramid by a plane parallel to the base, the smaller pyramid is similar to the original pyramid.



177. If any two lines, straight or curved, be drawn in a solid, and if the two corresponding lines be drawn in a similar solid, these four lines will be proportionals.

Thus in the two similar cones $ABCD$ and $abcd$—

Circumference BCD : circumference bcd = height AE : height ae.

PROPOSITION XLII.

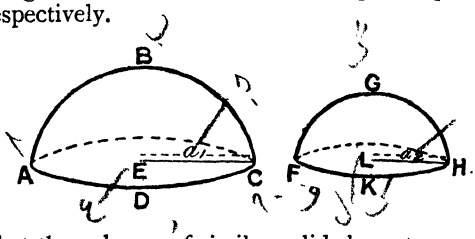
178. *Having given the lengths of two corresponding lines drawn in two similar solids, and the volume of one of these solids, to find the volume of the other solid.*

Let $ABCD$, $FGHK$ be two similar solids.

Let the corresponding lines EC and LH measure a_1 and a_2 of the same linear unit respectively.

Let the volume of the solid $FGHK$ measure V_2 of any solid unit.

It is required to find the volume of the solid $ABCD$ in terms of a_1 a_2 and V_2.



It can be proved that the volumes of similar solids have to one another the ratio of the cubes of the lengths of any two corresponding lines that may be drawn in them.

$\therefore$ vol. of solid $ABCD$: vol. of solid $FGHK = EC^3 : LH^3$

that is—

$$\text{Volume of solid } ABCD : V_2 = a_1^3 : a_2^3$$

Hence rule—

The volume of a solid is found by taking its ratio to the known volume of a similar solid, and equating it to the ratio of the cubes of known corresponding lengths in the two solids.

Or briefly—

Volume of first solid : volume of second solid = **ratio of the cubes of corresponding lengths in first solid and second solid**

$$V_1 : V_2 = a_1^3 : a_2^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

Hence—

$$a_1 : a_2 = \sqrt[3]{V_1} : \sqrt[3]{V_2} \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

ILLUSTRATIVE EXAMPLES.

179. *Example* 1.—Find what height must be cut off a pyramid to make one equal to four-fifths of the whole, the height of the pyramid being 20 ft.

The smaller pyramid and the pyramid from which it is cut off are similar solids . . . . . § 176.

Hence, if a_1 ft. = height of smaller pyramid, we have—

$$a_1 : a_2 = \sqrt[3]{V_1} : \sqrt[3]{V_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{§ 178.}$$

where $a_2 = 20$,
$V_1 = 4$,
$V_2 = 5$;

$$\therefore a_1 : 20 = \sqrt[3]{4} : \sqrt[3]{5}$$
$$a_1 = 20\sqrt[3]{\tfrac{4}{5}} = 4\sqrt[3]{100}$$

required height $= 4\sqrt[3]{100}$ ft.

Example 2.—A cubic foot of brass weighs 9000 ozs. : find the weight of a cube of brass whose diagonal is 12 in.

The diagonal of 1 cub. ft. measures $\sqrt{3}$ ft. . . . . § 119.

Hence, if the cube of 12-in. diagonal weigh w ozs., we have, by similar solids—

$$w : 9000 = 1^3 : (\sqrt{3})^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \S\ 178.$$

$$\therefore w = \frac{9000}{(\sqrt{3})^3}$$

$$= 1000\sqrt{3}$$

$$= 1732{\cdot}05$$

required weight = 1732·05 ozs.

Example 3.—Divide a cone into three equal parts by planes parallel to the base, and find the positions of these planes, the height of the cone being 30 in.

Let the parallel planes be distant x in. and y in. respectively from the apex of the cone.

Then, by similar solids—

$$x^3 : 30^3 = \tfrac{2}{3} : 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \S\ 178.$$

$$\therefore x = 30\sqrt[3]{\tfrac{2}{3}}$$

$$= 10\sqrt[3]{18}$$

$$\text{also } y^3 : 30^3 = \tfrac{1}{3} : 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \S\ 178.$$

$$\therefore y = 30\sqrt[3]{\tfrac{1}{3}}$$

$$= 10\sqrt[3]{9}$$

The planes are distant $10\sqrt[3]{18}$ in. and $10\sqrt[3]{9}$ in. respectively from the apex of the cone.

Example 4.—Supposing iron to be eight times the weight of oak, what will be the diameter of an iron ball whose weight is equal to that of a ball of oak 18 in. in diameter?

Let w lbs. be the weight of each ball,

Let v cub. in. and V cub. in. be the volumes of the balls of iron and oak respectively.

Then—

$$\frac{w}{v} = \frac{8w}{V}$$

$$\text{or } v : V = 1 : 8$$

But if a_1 in. = diameter of iron ball—

$$v : V = a_1^3 : 18^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \S\ 178.$$

$$\therefore a_1^3 : 18^3 = 1 : 8$$

$$a_1^3 = \frac{18^3}{8}$$

$$a_1 = \frac{18}{\sqrt[3]{8}}$$

$$= 9$$

Diameter of iron ball will be 9 in.

Example 5.—The area of the base of a cone is 25 sq. in.: find the area of the base of a similar cone whose volume is to that of the former as 8 : 1.

Let a_1 in. and a_2 in. be corresponding lengths in the two cones. Then—

$$a_1 : a_2 = \sqrt[3]{1} : \sqrt[3]{8} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \S\ 178.$$
$$\text{also } a_1 : a_2 = \sqrt{25} : \sqrt{A_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad \S\ 104.$$

where A_2 sq. in. = area of base of similar cone.

$$\therefore \sqrt{A_2} : \sqrt{25} = \sqrt[3]{8} : \sqrt[3]{1}$$
$$\therefore \sqrt{A_2} = 10$$
$$A_2 = 100$$

The base of the similar cone measures 100 sq. in.

Example 6.—The radii of the ends of a frustum of a right circular cone are 7 in. and 8 in. respectively, and the height is 3 in. : find the position of a plane parallel to the ends which shall divide the volume into two equal parts.

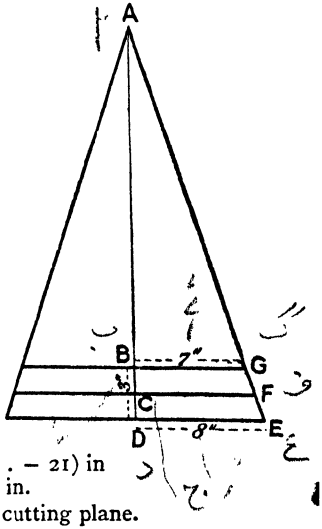
Complete the cone of which the frustum is a part.

If the figure represent a section of this cone through its axis, we have—

$$GB = 7 \text{ in.}$$
$$ED = 8 \text{ in.}$$
$$BD = 3 \text{ in.}$$

And, if $AB = x$ in.—

$$x : x + 3 = 7 : 8 \quad . \quad . \quad . \quad \S\ 66.$$
$$\therefore x = 21$$



Now, let FC indicate the position of the cutting plane, and let $AC = y$ in.

Also let the three cones ABG, ACF, $ADE = V_1$, V_2, V_3 cub. in. respectively.

Then, by similar solids—

$$V_1 : V_2 : V_3 = 21^3 : y^3 : 24^3 \quad \S\ 178.$$
but $V_3 - V_2 = V_2 - V_1$
$$\therefore 24^3 - y^3 = y^3 - 21^3$$
$$\therefore 2y^3 = 24^3 + 21^3 = 23{,}085$$
$$\therefore y^3 = 11{,}542{\cdot}5$$
$$\therefore y = 22{\cdot}599 \ldots$$
$$\therefore BC = (22{\cdot}599 \ldots - 21) \text{ in}$$
$$= 1{\cdot}599 \ldots \text{ in.}$$

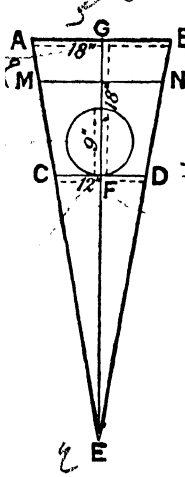
And this determines the position of the cutting plane.

Example 7.—A bucket has the form of a frustum of a cone, upper diameter $1\frac{1}{2}$ ft., lower 1 ft., depth $1\frac{1}{2}$ ft. If the bucket be half filled with water, to what height will the water rise if a sphere of 9 in. diameter be placed in it?

$$\text{Volume of bucket} = \frac{\pi \times 1\frac{1}{2}}{3}\{(\tfrac{3}{4})^2 + (\tfrac{1}{2})^2 + \tfrac{3}{8}\} \text{ cub. ft.} \quad . \quad \S\ 162.$$
$$= \frac{19\pi}{32} \text{ cub. ft.}$$
$$= 1026\pi \text{ cub. in.}$$
$$\therefore \text{volume of half the bucket} = 513\pi \text{ cub. in.}$$

Now complete the cone of which the bucket is a frustum.

If the figure represent a section of this cone through its axis, we have—



$AB = 18$ in.
$CD = 12$ in.
$GF = 18$ in.

And, if $EF = x$ in.—

x in. : $(x + 18)$ in. $= 12 : 18$ . . . . . § 66.
$\therefore x = 36$

$\therefore$ volume of cone $ECD = \frac{1}{3}.\pi.(6)^2 . 36$ cub. in. § 142.
$= 432\pi$ cub. in.

By putting in a sphere of 9 in. diameter, the water in the bucket will rise through a volume of $\frac{\pi \cdot 9^3}{6}$ cub. in. . . . . § 166.

$= \frac{243\pi}{2}$ cub. in.

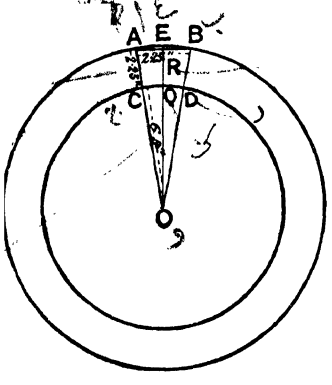
Hence, if the line MN indicate the level of the water after the insertion of the sphere, the frustum $CMND$ will measure $(513\pi + 121{\cdot}5\pi)$ cub. in.
$= 634{\cdot}5\pi$ cub. in.

Let y in. = the height to which the water will rise.
Then, by similar figures—

$(36 + y)^3 : 36^3 = 1066{\cdot}5\pi : 432\pi$ . . . . . § 178.
$36 + y = 18\sqrt[3]{\frac{79}{4}}$
$= 9\sqrt[3]{158}$
$= 48{\cdot}655$
$\therefore y = 12{\cdot}655$

The water rises to a height of 12·655 in.

Example 8.—A conical hole is made in a spherical shell, the vertex of the cone being in the centre of the shell. The outer diameter of the hole is 2·25 in., the inner diameter of the shell 13 in., and the thickness of the material 2·25 in. How much of its volume does the shell lose? ($\pi = 3{\cdot}1416$.)



If the figure represent a section through the centre of the sphere, we have—

$AB = 2{\cdot}25$ in.
$OC = 6{\cdot}5$ in.
$CA = 2{\cdot}25$ in.

Let $RE = h$ in.
Then—

$h(17{\cdot}5 - h) = (1{\cdot}125)^2$ § 75.
$h^2 - \frac{35}{2}h + \frac{81}{64} = 0$
$h = 0{\cdot}07262$

$\therefore$ volume of sector $OAEB = \frac{2}{3} . \pi r^2 h$ cub. in. . . § 174.

where $r = 8{\cdot}75$,

$h = 0{\cdot}07262$;

hence volume of sector $OAEB$ $= \frac{2}{3} \times 3{\cdot}1416 \times (8{\cdot}75)^2 \times 0{\cdot}07262$ cub. in.

Now, sector $OCQD$ and sector $OAEB$ are similar solids.

$\therefore$ volume of sector $OCQD$ $= \frac{2}{3} \times 3{\cdot}1416 \times (8{\cdot}75)^2 \times 0{\cdot}07262 \times \frac{(6{\cdot}5)^3}{(8{\cdot}75)^3}$ cub. in. § 178.

Hence volume cut away $= \frac{2}{3} \times 3{\cdot}1416 \times 0{\cdot}07262 \times \left\{(8{\cdot}75)^2 - \frac{(6{\cdot}5)^3}{8{\cdot}75}\right\}$ cub. in.

$= \frac{2}{3} \times 3{\cdot}1416 \times 0{\cdot}07262 \times 45{\cdot}1767$ cub. in.

$= \frac{2}{3} \times 3{\cdot}1416 \times 3{\cdot}28073$ cub. in.

$= 6{\cdot}87$ cub. in. nearly.

The shell loses 6·87 cub. in. of its volume nearly.

Examples—XXIX.

1. Find the ratio of the volumes of two spheres whose diameters are as 7 : 2.

2. The volume of one cube is eight times the volume of another : if the edge of the first is 2 ft. 6 in., what is the edge of the second ?

3. The heights of two similar pyramids are 6 in. and 7 in. respectively : if the volume of the first is 50 cub. in., find the volume of the second.

4. If a sphere 5 in. in diameter weigh 8 lbs., find the weight of another sphere of the same material 8 in. in diameter.

5. The height of a cone is 5 ft. : find the height of a similar cone twenty-seven times its volume.

6. A pyramid whose height is 20 in. is cut into two equal parts by a plane parallel to the base : find the height of each part.

7. The volumes of two similar prisms are as 343 : 125 : find the ratio of their heights.

8. The weights of two cannon balls are as 1000 : 729 : find the ratio of their radii.

9. The weights of two similar cones are 5832 ozs. and 4913 ozs. respectively : if the height of the first is 9 ft., find the height of the second.

10. A pyramid is cut by a plane parallel to the base and midway between the vertex and the base : find the ratio of the volume of the frustum to the volume of the whole pyramid.

11. The diameters of the ends of a frustum of a cone are respectively 12 ft. and 8 ft., and the height of the frustum is 4 ft. ; the frustum is divided into two equal parts by a plane parallel to the ends : find the distance of the plane from the smaller end.

12. Divide a pyramid into three equal parts by planes parallel to the base, and find the altitudes of these parts, the height of the pyramid being 12 ft.

Examination Questions—XXIX.

A. *Bombay University, Diploma of Agriculture : Second Exam.*

1. A bucket is in the shape of a frustum of a cone ; the diameter at the bottom is 9 in., and at the top 1 ft. ; the height is 14 in. : when it is half filled with water, find the depth of the water.

B. *Punjab University: First Exam. in Civil Engineering.*

2. Each edge of a cube is diminished by one-tenth of its length: by how much is the volume diminished?

C. *Calcutta University: F.E. Exam.*

3. If a right circular cone be divided into three parts by two planes parallel to the base trisecting the axis, compare the three volumes into which it is divided.

D. *Sibpur Apprentice Dept.: Monthly.*

4. The base of a pyramid is $7\frac{1}{2}$ in. square: required the base of a similar pyramid whose volume is to that of the former as 111 : 11.

5. Compare the volumes of two similar cones whose circumferences are respectively 15 ft. and 12 ft.

6. If a rectangular parallelopiped has its length, its breadth, and its depth respectively a quarter as large again as another rectangular parallelopiped, show that the first is nearly twice as large again as the second.

E. *Sibpur Apprentice Dept.: Final.*

7. A pyramid is cut into two pieces by a plane parallel to the base, midway between the vertex and the base: show that one piece is equal to seven times the other.

F. *Roorkee Upper Subordinate: Entrance.*

8. The diameters of the ends of a frustum of a cone are respectively 20 ft. and 16 ft., and the height of the frustum is 5 ft.; the frustum is divided into two equal parts by a plane parallel to the ends: find the distance of the plane from the smaller end.

9. The diameters of the ends of a frustum of a cone are respectively 20 ft. and 16 ft., and the height of the frustum is 5 ft.; the frustum is divided into three equal parts by planes parallel to the ends: find the distances of the planes from the smaller end.

10. Divide a cone whose height is 81 in. into three equal segments by planes parallel to the base, and find the perpendicular height of each portion.

G. *Roorkee Upper Subordinate: Monthly.*

11. If the model of a steam-engine weigh 80 lbs., find the weight of the engine made of the same material as the model, but of nine times its linear dimensions.

12. The weights of two globes are as 9 : 25; the weights of a cubic inch of the substances are as 15 : 9: compare the diameters of the globes.

H. *Roorkee Engineer: Final.*

13. Show that the volume of a sphere whose radius is 6 in. is equal to the sum of the volumes of the spheres whose radii are 3 in., 4 in., and 5 in.

14. Divide a cone into three equal parts by sections parallel to the base, and find the altitudes of these parts, the height of the cone being 20 in.

Additional Examination Questions—XXIX.

15. Two cubical blocks of stone together contain 1,791,153 cub. in., and the edge of the less is to that of the greater as 3 : 4. Find the edge of each. (Allahabad University: Intermediate.)

16. A conical glass, 3 in. in diameter at the top and 4 in. deep, is half filled with water: how much water does it contain, and how high does it come in the glass? (Roorkee Upper Subordinate: Monthly.)

PART III.

CHAPTER XXX.

ON SOLIDS BOUNDED BY PLANE SURFACES.

180. The area of the whole surface of a solid is equal to the sum of the areas of its bounding surfaces.

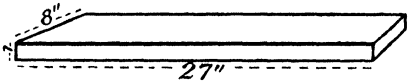
If these bounding surfaces are *plane figures*, the area of each bounding surface can be found by one or other of the rules arrived at in plane mensuration, and no further investigation is necessary.

The following solids are bounded by plane surfaces :—

Rectangular solids.
Prisms.
Pyramids.
Wedges.
Oblique frusta of prisms.
Prismoids.
Frusta of pyramids.

ILLUSTRATIVE EXAMPLES.

181. *Example* 1.—A plate of metal 1 in. thick, 8 in. broad, and 27 in. long, is melted into a cube. Find the difference in the surfaces of the two solids.



Surface of metal plate $= 2(8 \times 27 + 27 \times 1 + 8 \times 1)$ sq. in. § 8.
$= 502$ sq. in.

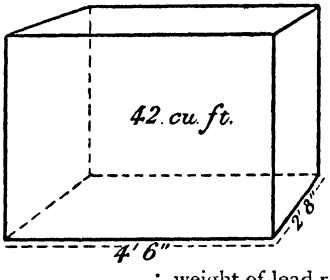
Volume of metal plate $= (8 \times 27 \times 1)$ cub. in. . . . . § 115.
$= 216$ cub. in.

$\therefore$ edge of cube $= \sqrt[3]{216}$ in. . . . . § 117.
$= 6$ in.

$\therefore$ surface of cube $= 6 \times (6 \times 6)$ sq. in. . . . . § 9.
$= 216$ sq. in.

hence difference in the surfaces of the two solids $= (502 - 216)$ sq. in.
$= 286$ sq. in.

Example 2.—A cistern open at the top is to be lined with sheet lead which weighs $5\frac{1}{2}$ lbs. to the square foot; the cistern is 4 ft. 6 in. long, 2 ft. 8 in. wide, and holds 42 cub. ft.: find the weight of lead required.



If x ft. = depth of cistern—

$$x \times 4\tfrac{1}{2} \times 2\tfrac{2}{3} = 42 \quad \S\ 115.$$

or $x = 3\frac{1}{2}$

that is, depth of cistern $= 3\frac{1}{2}$ ft.

$\therefore$ quantity of lead necessary to line the cistern—

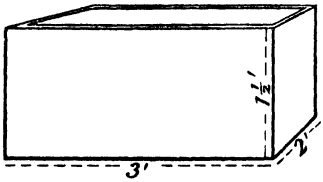
$= (4\frac{1}{2} \times 2\frac{2}{3} + 2 \times 4\frac{1}{2} \times 3\frac{1}{2} + 2 \times 2\frac{2}{3} \times 3\frac{1}{2})$ sq. ft. . . . . § 8.

$= 62\frac{1}{6}$ sq. ft.

$\therefore$ weight of lead required $= 62\frac{1}{6} \times 5\frac{1}{2}$ lbs.

$= 341\frac{1}{12}$ lbs.

Example 3.—A box, without a lid, and made of wood 1 in. thick, requires painting inside and out. Its exterior length, breadth, and depth are 3, 2, and $1\frac{1}{2}$ ft. respectively. How many superficial feet of paint will be required for each coat?



Outside surface—

$= \{3 \times 2 + 2(3+2) \times 1\frac{1}{2} + 2(3 + 1\frac{5}{6}) \times \frac{1}{12}\}$ sq. ft. . . . . § 8.

$= (6 + 15 + \frac{29}{36})$ sq. ft.

$= 21\frac{29}{36}$ sq. ft.

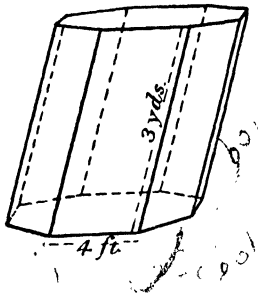
Inside surface $= \{2\frac{5}{6} \times 1\frac{5}{6} + 2(2\frac{5}{6} + 1\frac{5}{6}) \times 1\frac{5}{12}\}$ sq. ft. . § 8.

$= (\frac{187}{36} + 11\frac{19}{36})$ sq. ft.

$= 18\frac{5}{12}$ sq. ft.

$\therefore$ each coat will require $40\frac{2}{3}$ sq. ft. of paint.

Example 4.—Find the area of the whole surface of an oblique prism having a regular octagonal base whose side is 4 ft., the lateral edges being 3 yds., and the perimeter of a section perpendicular to them 9 yds. 1 ft.



Area of the two end surfaces of the prism—

$= 2 \times 2(4)^2(1 + \sqrt{2})$ sq. ft. . . § 45.

$= 64(1 + \sqrt{2})$ sq. ft.

Now, since the side faces of the prism are all parallelograms having a common length which is equal to a lateral edge—

$\therefore$ area of lateral surface $\Big\}$ $= \Big\{$ perimeter of perpendicular section $\times$ lateral edge

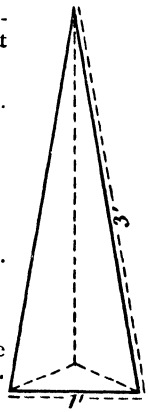
$= (28 \times 9)$ sq. ft.

$= 252$ sq. ft.

hence the area of the whole surface $= \{252 + 64(1 + \sqrt{2})\}$ sq. ft.

$= 406{\cdot}509$ sq. ft.

Example 5.—A pyramid has for its base an equilateral triangle of which each side is 1 ft., and its slant edge is 3 ft. : find its whole surface.



Slant surface of pyramid $= 3 \times \frac{c}{4}\sqrt{4a^2 - c^2}$ sq ft. § 24.

where $c = 1$,
$a = 3$;

$\therefore$ slant surface of pyramid $= 3 \times \frac{1}{4}\sqrt{36 - 1}$ sq. ft.
$= 4{\cdot}437$ sq. ft.

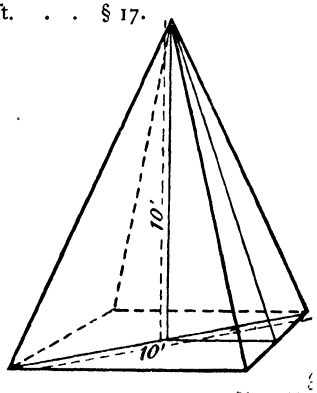
base of pyramid $= \frac{\sqrt{3}}{4}$ sq. ft. . . . § 21.
$= 0{\cdot}433$ sq. ft.

$\therefore$ whole surface $= 4{\cdot}87$ sq. ft. nearly

Example 6.—A right pyramid 10 ft. high has a square base of which the diagonal is 10 ft. : find its slant surface.

Since the diagonal of the square base = 10 ft.—

$\therefore$ each side of the square base $= \frac{10}{\sqrt{2}}$ ft. . . § 17.



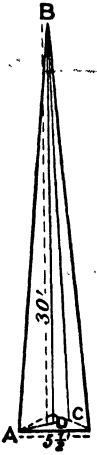
And since the height of the pyramid = 10 ft.—

$\therefore$ slant height of pyramid $= \sqrt{(10)^2 + \left(\frac{5}{\sqrt{2}}\right)^2}$ ft. § 16.

$= \frac{15}{\sqrt{2}}$ ft.

$\therefore$ slant surface of pyramid $= 4 \times \frac{1}{2} \times \frac{10}{\sqrt{2}} \times \frac{15}{\sqrt{2}}$ sq. ft. § 20.

$= 150$ sq. ft.



Example 7.—Find the whole surface of a triangular pyramid, each side of the base being $5\frac{1}{2}$ ft., and perpendicular height 30 ft.

A section of the pyramid through a slant edge AB, and through C the middle point of a side of the base, will contain BD the perpendicular from the vertex B on to the base.

And the point D, the foot of this perpendicular, is so situated that—

$$DC = \tfrac{1}{3}AC \quad \text{§ 93.}$$

$$= \tfrac{1}{3} \cdot 5\tfrac{1}{2} \cdot \frac{\sqrt{3}}{2} \text{ ft.} \quad \text{§ 17.}$$

$$= \frac{11}{4\sqrt{3}} \text{ ft.}$$

hence—

$$BC = \sqrt{(30)^2 + \left(\frac{11}{4\sqrt{3}}\right)^2} \text{ ft.} \quad \text{§ 16.}$$

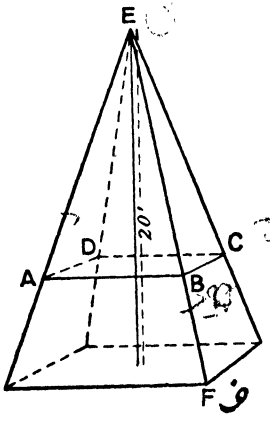
$$= 30{\cdot}0419 \text{ ft.}$$

$\therefore$ area of slant surface $= 3 \times \frac{1}{2} \times \frac{11}{2} \times 30{\cdot}0419$ sq. ft. . § 20.
$= 247{\cdot}843$ sq. ft.

also area of base $= (5\frac{1}{2})^2 . \frac{\sqrt{3}}{4}$ sq. ft. . . . . § 21.
$= 13{\cdot}098$ sq. ft.

hence area of whole surface $= 260{\cdot}94$ sq. ft.

Example 8.—The altitude of a square pyramid is 20 ft.: find the perpendicular distance from the vertex of a horizontal plane which will cut the area of the slant surface into two equal parts.



Let *ABCD* in the figure indicate the position of the cutting plane, and let x ft. = the perpendicular distance of the cutting plane from the vertex of the pyramid.

Then, by similar figures—

$x : 20 = EB : EF$ . . § 177.

But, by similar figures—

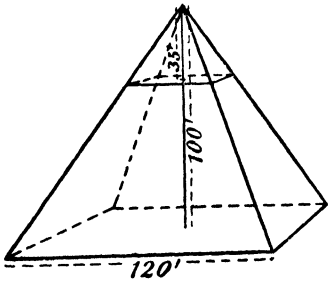
$EB : EF = \sqrt{1} : \sqrt{2}$ . . § 104.

$\therefore x : 20 = \sqrt{1} : \sqrt{2}$

or $x = 10\sqrt{2}$
$= 14{\cdot}1421$

Required distance is $14{\cdot}1421$ ft.

Example 9.—The perpendicular height of a pyramid is 100 ft., and its base is 120 ft. square; 35 ft. of perpendicular height is removed from the summit: what will then be the whole exposed superficial area of the remainder?



The remainder will be a frustum of the pyramid.

The top of this frustum will be a square.

Let each side of the top $= x$ ft.

Then, by similar figures—

$x : 120 = 35 : 100$ § 177.
or $x = 42$

Again—

Slant height of pyramid $= \sqrt{(60)^2 + (100)^2}$ ft. . . § 16.
$= 20\sqrt{34}$ ft.

And, if y ft. = perpendicular distance between the parallel sides of each slant face of the frustum, we have, by similar figures—

$y : 20\sqrt{34} = 65 : 100$ . . . . . § 177.
or $y = 13\sqrt{34}$

hence area of slant surface $= 4 \times \{\frac{1}{2}(42 + 120) \times 13\sqrt{34}\}$ sq. ft. § 39.
$= 24{,}559{\cdot}96$ sq. ft.
also area of top $= (42)^2$ sq. ft. . . . . § 9.
$= 1764$ sq. ft.
hence whole exposed superficial area $= 26{,}323{\cdot}96$ sq. ft.

Examples—XXX.

Rectangular Solids.

Find the areas of the whole surfaces of rectangular solids having the following dimensions :—

1. 3 ft. 6 in., 2 ft. 9 in., 2 ft. 6 in.

2. 4 ft. 7 in., 3 ft. 10 in., 3 ft. 8 in.

3. 2 yds. 2 ft. 9 in., 2 yds., 1 yd. 2 ft. 7 in.

Find the areas of the whole surfaces of cubes having the following edges :—

4. 3 ft. 7 in.

5. 1 yd. 2 ft. 6 in.

6. 2 yds. 2 ft. 8 in.

7. Find the cost of painting the outside of a rectangular box whose length is 5 ft. 4 in., breadth 4 ft. 6 in., and height 4 ft. 3 in., at 3*d.* per square yard.

8. Find the edge of a cube whose surface has the same area as that of a rectangular solid of the following dimensions: length 10 ft., breadth 7 ft., depth 6 ft.

9. Find the area of the surface of a cube whose volume is equal to the volume of a rectangular solid of the following dimensions : length 9 ft., breadth 7 ft., depth 5 ft.

10. The base of a rectangular solid is a square, and its height is twice its length : if its volume is 2000 cub. in., find the area of its surface.

11. The base of a rectangular box is a square, and its internal depth is half its internal length : if the capacity of the box is 4000 cub. in., and if the box is without a lid, find its internal surface.

12. A rectangular box is made of wood 1 in. thick ; the box is fitted with a lid : if the internal dimensions are 4 ft. 6 in., 3 ft. 8 in., and 3 ft. 5 in., find the area of the whole external surface.

13. Find to the hundredth part of one inch the edge of a cube whose surface measures 10 sq. ft.

14. Find to the nearest penny the cost of lining the sides and bottom of a rectangular cistern 10 ft. 4 in. long, 6 ft. 2 in. broad, 6 ft. 8 in. deep, with sheet lead which costs 25*s.* per cwt. and weighs 8 lbs. to the square foot.

15. Verify the statement that to make a rectangular box, *with a lid*, of a given capacity from a minimum of material, the cube is the most advantageous shape.

16. Verify the statement that to make a rectangular box, *without a lid*, of a given capacity from a minimum of material, the most advantageous shape is that in which the height is half the length and the base is a square.

Prisms.

17. Find the area of the whole surface of a right triangular prism whose height is 36 ft., and the sides of whose base are 51, 37, and 20 ft. respectively.

18. Find the area of the whole surface of a right triangular prism whose height is 7 yds., and the sides of whose base are 25, 39, and 56 ft. respectively.

19. Find the area of the whole surface of a right prism whose height is 6 ft., and whose base is a regular hexagon of side 2 ft.

20. Find the area of the whole surface of a right prism whose height is 10 in., and whose base is a regular octagon of side 1 in.

21. Find the area of the whole surface of a right prism whose height is 2 ft. 3 in., and whose base is an equilateral triangle on a side of 1 ft.

22. Find the cost of polishing the side surfaces of a right prism whose height is 5 ft. 3 in., and whose base is a regular nonagon of side 1 ft. 8 in. at 10*d.* per square foot.

23. The base of a right prism is a triangle whose sides measure 13, 14, 15 in. respectively: find the height of the prism if the whole surface measures 4 sq. ft. 12 sq. in.

24. Find the area of the whole surface of an oblique prism having a square base whose side is 8 in., the lateral edges being 2 ft. 3 in. and the perimeter of a section perpendicular to them 3 ft. 2 in.

25. The area of the whole surface of a right prism on an equilateral triangle as base is $25\sqrt{3}$ sq. in., and the area of the lateral surface is equal to the area of the two ends: find the height of the prism.

Pyramids.

26. Find the area of the whole surface of a right pyramid whose base is a square of side 2 ft. 6 in., and whose slant height is 2 ft. 9 in.

27. Find the area of the whole surface of a regular tetrahedron whose edge measures 5 ft.

28. Find the area of the whole surface of a right pyramid whose base is a regular hexagon of side 10 in., and whose slant height is 1 ft.

29. Find the area of the whole surface of a right pyramid whose base is a square of side 40 in., and whose other edges each measure 20·5 in.

30. Find the area of the whole surface of a right pyramid whose base is a square of side 16 in., and whose height is 29 in.

31. Find the height of a right pyramid whose whole surface measures $270\sqrt{3}$ sq. in., and whose base is an equilateral triangle of side $10\sqrt{3}$ in.

32. Find the cost of polishing the whole surface of a right pyramid whose slant height is 14 in., and whose base is a regular octagon of side 1 in. at 6*d.* per square inch.

33. Find the slant surface of a right pyramid whose height is 65 in., and whose base is a regular hexagon of side $48\sqrt{3}$ in.

34. A pyramid, on a square base, has four equilateral triangles for its four other faces, each edge being 9 in.: find the whole surface.

35. The slant edge of a right regular hexagonal pyramid is 65 in., and the height is 56 in.: find the area of the base.

Wedges.

36. The length of the base of a wedge is 70 in., its breadth is 56 in.; the edge of the wedge is 86 in.: if the other sides of the trapezoidal faces are each 32·5 in., find the area of the whole surface of the wedge.

37. The length of the base of a wedge is 60 in. and the breadth 24 in.; the height of the wedge is 9 in. and the edge 52 in. If the ends of the wedge are equally inclined to the base, find the area of the trapezoidal sides.

Oblique Frusta of Prisms.

38. The base of a right prism is a regular heptagon of side 3 ft.; an oblique frustum is obtained by cutting off a portion of this prism so that the sum of the seven parallel edges is 38 ft. 6 in.: find the area of the side faces of the frustum.

39. The cross-section of an oblique prism is a regular pentagon of side 1 ft. 6 in.; an oblique frustum is obtained by cutting off a portion of this prism so that the sum of the five parallel edges is 22 ft. 6 in.: find the area of the side faces of the frustum.

Prismoids.

40. The ends of a prismoid are rectangles whose corresponding dimensions are 56 ft. by 28 ft. and 20 ft. by 15 ft. respectively; each of the remaining edges is 42 ft. 6 in.: find the area of the whole surface.

41. A prismoid has one end in the form of a square of side 16 in., the other in the form of a regular octagon of side 8 in., four sides of the octagon being parallel to the sides of the square: if the other edges of the prismoid each measure 8·5 in., find the area of its lateral surface.

Frusta of Pyramids.

42. The ends of a frustum of a pyramid are equilateral triangles of sides 9 ft. and 13 ft. respectively: if the distance between the parallel sides of each trapezoidal face is 6 ft. 4 in., find the area of the *slant* surface of the frustum.

43. The ends of a frustum of a pyramid are squares of sides 2 ft. 6 in. and 3 ft. 9 in. respectively: if the distance between the parallel sides of each trapezoidal face is 3 ft. 4 in., find the area of the *whole* surface of the frustum.

44. Find to the nearest penny the cost of polishing the whole surface of a frustum of a pyramid whose ends are regular hexagons of sides 3 ft. and 4 ft. respectively, and whose slant height is 1 ft. 6 in., at 3*d.* per square inch.

45. The ends of a frustum of a pyramid are squares of sides 4 in. and 10 in. respectively; the height is 4 in.: find the area of the whole surface.

Examination Questions—XXX.

(Take $\pi = \frac{22}{7}$.)

Rectangular Solids.

A. *Roorkee Engineer: Entrance.*

1. Find the cost of lining the sides and bottom of a rectangular cistern 12 ft. 9 in. long, 8 ft. 3 in. broad, 6 ft. 6 in. deep, with sheet lead which costs £1 8*s.* per cwt., and weighs 8 lbs. to the square foot.

2. The diagonal of the base of a cube is 2 ft.: find its volume and whole surface. What will be the least possible loss of material in turning it into a ball?

3. How many coins $\frac{3}{4}$ in. in diameter and $\frac{1}{8}$ in. thick must be melted down to form a cube whose surface measures 54 sq. in.?

B. *Roorkee Upper Subordinate: Entrance.*

4. A cube contains 650·962 ft.: find the cost of covering it with material 2 ft. 2 in. wide at 9*d.* per yard.

C. *Roorkee Engineer: Final.*

5. The adjacent edges of a rectangular box are 3·428571 in., 5·142857 in. and 10·285714 in.: find the cost of gilding its exterior at $1\frac{1}{2}$*d.* per square inch.

6. The cost of a cube of metal at £3 10*s.* 4*d.* per cubic inch is £1206 4*s.* 4*d.*: find the cost of gilding it over at $\frac{1}{2}$*d.* per square inch.

7. Compare the area of a section of a cube through two opposite edges with the whole surface of the cube.

8. If a punkah frame 15 ft. long, $2\frac{1}{2}$ ft. wide, and $1\frac{1}{2}$ in. thick is to be covered with cloth, how many running feet of cloth will be required, supposing the cloth to be 2 ft. wide?

Prisms.

A. *Bombay University, Diploma in Agriculture: Second Exam.*

9. What is the area of the surface of an oblique prism having a regular hexagonal base whose side is 10 in., the lateral edges being 20 ft., and the perimeter of a section perpendicular to them $4\frac{1}{2}$ ft.?

B. *Superior Accounts.*

10. The area of the whole surface of a regular octagonal prism is 2070 sq. ft., and the area of the lateral surface is twice the area of the top: find the length of each side of the base.

Pyramids.

A. *Bombay University, Diploma of Agriculture: Second Exam.*

11. A pyramidal roof 12 ft. high, standing on a rectangular base 18 ft. by 32 ft., is covered with slates which cost 18*s.* 9*d.* per hundred, and each of which has an exposed surface of 12 in. by 9 in.: find the cost.

B. *Bombay University: L.C.E. Second Exam.*

12. Show how to find the area of the surface of a regular pyramid.

C. *Punjab University: First Exam. in Civil Engineering.*

13. Find the area of the inclined surface of a square pyramid, each side of the base being 3 ft., and the slant height 15 ft.

D. *Calcutta University: F.E. Exam.*

14. Find the slant surface of a right pyramid of height h on a regular base of side a and of n sides.

E. *Roorkee Engineer: Entrance.*

15. It is desired to cover a piece of ground 21 ft. square by a pyramidal tent 14 ft. in perpendicular height: find the cost of the requisite canvas at 5 annas a square yard.

16. Find the area of the whole surface of a triangular pyramid contained by four equilateral triangles, a side of each being 10 ft.

17. The Pyramid of Cheops is 750 ft. square and 450 ft. high. What would it cost to restore the surface to its original splendour with polished granite at the rate of £1 per superficial foot?

F. *Roorkee Upper Subordinate: Entrance.*

18. A pyramid has for its base an equilateral triangle, of which each side is 2 ft., and its slant edge 6 ft.: find its exposed surface.

19. Find the area of the whole surface of a pyramid on a triangular base, having its other faces equal, each side of the base is 1·45 in., and the slant edge of the pyramid is 2·68 in.

20. The slant edge of a hexagonal spire 75 ft. high is 77 ft.: find the cost of painting at Rs.4 per 100 superficial feet.

G. *Roorkee Engineer: Final.*

21. Find the cost of canvas required for a single-pole tent 12 ft. square with walls 6 ft. high. Roof slopes at an angle of 45°, and projects 3 ft. beyond the walls all round; canvas cloth 2 ft. 3 in. wide costs 14 annas per yard.

Prismoids.

Roorkee Engineer: Entrance.

22. A prismoid has one end in the form of an equilateral triangle of side 2 ft., the other end in the form of a regular hexagon of side 1 ft., three sides of the hexagon being parallel to the three sides of the other end; the height is 3 ft.: find the area of its surface.

Frusta of Pyramids.

A. *Bombay University, Diploma of Agriculture: Second Exam.*

23. Find the surface of the frustum of a square pyramid, each side of the base or greater end being 3 ft. 4 in., each side of the top or lesser end being 2 ft. 2 in., and each of the edges of the frustum being 10 ft.

B. *Madras University: B.E. Exam.*

24. The ends of a frustum of a pyramid are hexagons with sides of 6 ft. and 4 ft. respectively; the slant height is 10 ft.: find the surface.

C. *Roorkee Engineer: Entrance.*

25. The area of the surface of a frustum of a square pyramid is 100 sq. ft., the perimeter of the base is 13 ft. 4 in., and the slant height is 10 ft.: find the area of the top.

Additional Examination Questions—XXX.

26. A frustum of a regular pyramid has square ends; the edge of the lower end is 10 in., and that of the upper end is 5 in., and the height of the frustum is $7\frac{1}{2}$ in.: find the length of a slant edge of the frustum and the area of the slant faces. (Roorkee Upper Subordinate: Entrance.)

27. A box with a lid is made of planking $1\frac{1}{2}$ in. thick: if the external dimensions be 3 ft. 6 in., 2 ft. 6 in., and 1 ft. 9 in., find exactly how many square feet of planking are used in the construction. (Punjab University: First Exam. in Civil Engineering.)

28. A rectangular block of stone is to be polished on all its faces except that resting on the ground. By putting three different faces down, the total surfaces to be treated are found to be 412, 394, and 404 sq. ft. respectively: what are the dimensions of the block? (Roorkee Engineer: Final.)

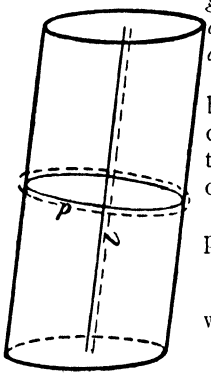
29. A prismoid has one end in the form of an equilateral triangle of side 3 ft., the other end in the form of a regular hexagon of side 2 ft., three sides of the hexagon being parallel to the three sides of the other end. The height is 3 ft.: find the area of its surface. (Roorkee Upper Subordinate: Entrance.)

CHAPTER XXXI.

ON CYLINDERS AND RINGS.

PROPOSITION XLIII.

182. *To find the area of the curved surface of any cylinder, having given the length of the cylinder and the perimeter of its cross-section by a plane perpendicular to its axis.*



We have seen (§ 125) that a cylinder may be defined as the limit of a prism, the number of whose sides is indefinitely increased while the breadth of each side is indefinitely diminished.

Now the area of the side surfaces of any prism is evidently equal to—

$$p \times l \text{ square units}$$

where p linear units = perimeter of a cross-section of the prism by a plane perpendicular to the axis,

l linear units = length of the prism;

$$\therefore \text{area of curved surface of any cylinder} = \text{perimeter of cross-section} \times \text{length} = pl \text{ square units}$$

Hence rule—

Multiply the number of any linear unit in the length of a cylinder by the number of the same linear unit in the perimeter of a cross-section by a plane perpendicular to the axis, then the product will give the number of the corresponding square unit in the area of the curved surface.

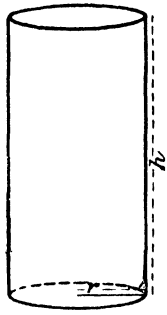
Or briefly—

Curved surface of cylinder = perimeter of cross-section × length

$$S = pl$$

PARTICULAR CASE.

183. Right circular cylinder.



Here the base of the cylinder is a cross-section by a plane perpendicular to the axis; and the length of the cylinder is the same as its height.

$\therefore$ curved surface of a right circular cylinder $= \left\{\begin{array}{l}\text{circumference of base} \\ \quad \times \text{ height}\end{array}\right.$
$= C \times h$ square units

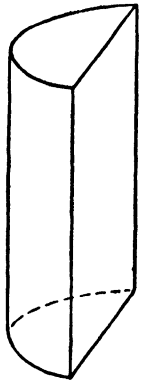
Hence—

Whole surface of right circular cylinder $= \left\{\begin{array}{l}2\pi r(r+h) \text{ square} \\ \quad \text{units}\end{array}\right.$

where r linear units = radius of base,
h linear units = height.

184. A segment of a right circular cylinder by a plane parallel to the axis is a prism in the wider sense of the term (see § 126). Hence the area of its side surfaces is determined by the formula—

$$S = p \times l \text{ square units}$$



where p linear units = perimeter of one end of segment,
l linear units = length of segment.

The perimeter of an end is determined by the formulæ for the arc of a circle (see §§ 79, 81).

The areas of the ends are determined by the formulæ for the segment of a circle (see §§ 88, 90).

Rings.

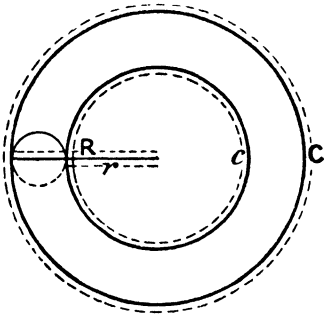
185. A cylindrical ring has been roughly described as a right circular cylinder bent round in a circle until its ends meet. Since, in bending the cylinder to form a ring, the inner portion is as much contracted as the outer portion is expanded, the surface of the ring may be seen to be the same as the surface of the original cylinder. Hence the surface of a cylindrical ring is equal to the surface of a right circular cylinder whose base is the same as the cross-section of the ring, and whose height is equal to the length of the ring.

That is—

Surface of a cylindrical ring $= \left\{\begin{array}{l}\text{perimeter of cross-section} \times \text{length of} \\ \quad \text{ring}\end{array}\right.$

$$S = p \times l$$

The same reasoning applies to the case of any ring whose cross-section is a figure symmetrical about a line in its own plane perpendicular to the plane of the ring (see figures to § 133).



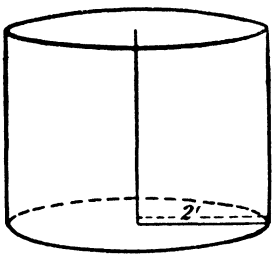
186. In the particular case of the cylindrical ring, the following formulæ may be easily verified :—

$$S = \pi^2(R^2 - r^2)$$
$$S = \tfrac{1}{4}(C^2 - c^2)$$

where S = surface area,
R and r = outer and inner radii respectively,
C and c = corresponding circumferences.

ILLUSTRATIVE EXAMPLES.

187. *Example* 1.—The radius of a right circular cylinder is 2 ft., and the area of its whole surface $20 \times \pi$ sq. ft. : what is the volume of a cone on the same base and of the same height?



If h ft. = height of cylinder, we have—

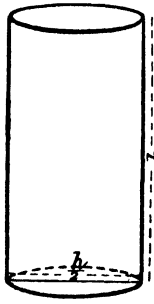
$$2\pi r(h + r) = 20\pi \quad . \quad . \quad . \quad . \quad \S\ 183.$$

where $r = 2$;

$$\therefore 4\pi(h + 2) = 20\pi$$
$$h = 3$$

hence vol. of required cone $= \tfrac{1}{3} . \pi . (2)^2 . 3$ cub. ft. § 142.
$= 4\pi$ cub. ft.

Example 2.—The whole surface of a right circular cylinder is 7 sq. ft. 37 sq. in., and the diameter of the base is half the height : find the height.



If h in. = height, we have—

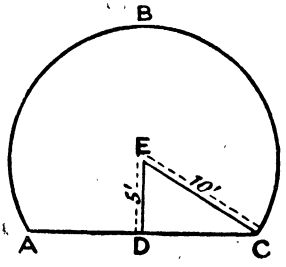
$$2\pi r(h + r) = 1045 \quad . \quad . \quad . \quad . \quad \S\ 183.$$

where $r = \dfrac{h}{4}$

$$\therefore 2\pi . \frac{h}{4}\left(h + \frac{h}{4}\right) = 1045$$
$$h^2 = 532$$
$$h = 23{\cdot}06$$

height of cylinder = 23·06 in.

Example 3.—A right circular cylinder, whose height is 12 ft., and the radius of whose base is 10 ft., is cut into two segments by a plane parallel to the axis and distant 5 ft. from it: find the area of the whole surface of the larger segment. ($\pi = 3{\cdot}1416$.)



Let the figure represent one end of the larger segment of the cylinder, so that AC indicates the position of the cutting plane.

Now, because—

$EC = 10$ ft.
and $ED = 5$ ft.
and $\angle EDC$ is a right angle—

$$\therefore \angle DEC = 60^\circ$$
$$\text{and } DC = \frac{10\sqrt{3}}{2} \text{ ft.} \quad \ldots\ldots \S 17.$$
$$\text{or } AC = 10\sqrt{3} \text{ ft.}$$

Again—

$$\text{Arc } ABC = \tfrac{240}{360} \times 2 . \pi . 10 \text{ ft.} \quad \ldots \S 79.$$
$$= \frac{40\pi}{3} \text{ ft.}$$
$$\therefore \text{perimeter of segment } ABCD = \left(10\sqrt{3} + \frac{40\pi}{3}\right) \text{ ft.}$$
$$= \text{perimeter of one end of the segment of the cylinder}$$

Also—

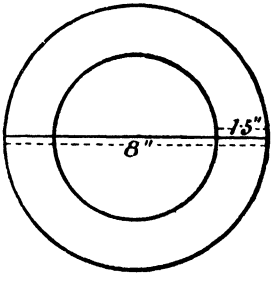
$$\text{Area of segment } ABCD = (\tfrac{240}{360} \times \pi . 10^2 + \tfrac{1}{2} . 5 . 10\sqrt{3}) \text{ sq. ft.} \quad \ldots \S\S 84, 20.$$
$$= \left(\frac{200\pi}{3} + 25\sqrt{3}\right) \text{ sq. ft.}$$

But area of whole surface of segment = (perimeter of an end × height of segment) + area of two ends . . . . . § 182.

$$\text{hence area of whole surface of segment} = \left\{\left(10\sqrt{3} + \frac{40\pi}{3}\right) \times 12 + 2\left(\frac{200\pi}{3} + 25\sqrt{3}\right)\right\} \text{ sq. ft.}$$

$= \{710{\cdot}502 + 505{\cdot}482\}$ sq. ft.
$= 1215{\cdot}98$ sq. ft.

Example 4.—The thickness of a solid cylindrical ring is 1·5 in., and its outside diameter 8 in.: find its surface. ($\pi = 3{\cdot}1416$.)



Surface of ring = $p \times l$ sq. in. . § 185.
where $p = 1{\cdot}5 \times \pi$ . . . . . § 69.
$l = \pi(8 - 1{\cdot}5)$ . . . . . § 69.
$\therefore$ surface of ring = $\pi . (1{\cdot}5) . \pi . (6{\cdot}5)$ sq. in.
$= 96{\cdot}229$ sq. in.

Examples—XXXI.

(Take $\pi = \frac{22}{7}$, unless otherwise stated.)

Find the areas of the *curved* surfaces of the following right circular cylinders :—

1. Radius of base 1 ft. 9 in., height 6 ft.

2. Diameter of base 2 ft. 3 in., height 7 ft. 2 in.

3. Circumference of base 5 ft. 2 in., height 1 ft. 7 in.

Find the areas of the *whole* surfaces of the following right circular cylinders :—

4. Radius of base 10 in., height 2 ft. 3 in.

5. Diameter of base 1 ft. 4 in., height 2 ft. 10 in.

6. Circumference of base 11 ft., height 3 ft. 2 in.

7. The curved surface of a right circular cylinder is 1 sq. ft. 54 sq. in., and the height is 3 in. : find the radius of the base.

8. The curved surface of a right circular cylinder is 1 sq. ft. 76 sq. in., and the diameter of the base is 10 in. : find the height.

9. The whole surface of a right circular cylinder is $5\frac{1}{2}$ sq. ft., and the radius of the base is 6 in. : find the height.

10. The whole surface of a right circular cylinder is 2 sq. ft. 20 sq. in., and the height is $10\frac{1}{2}$ in. : find the radius of the base.

11. The whole surface of a right circular cylinder is 8 sq. ft. 80 sq. in., and the height is three times the radius of the base : find the radius of the base.

12. Find to the nearest penny the cost of polishing the whole surface of a right circular cylinder whose height is 12 ft., and the radius of whose base is 1 ft. 3 in., at the rate of 10*d.* per square foot.

13. The volume of a right circular cylinder is 1100 cub. in., and the radius of its base is 5 in. : find the area of its curved surface.

14. What is the proportion between the height of a right circular cylinder and the radius of its base when the area of the two ends is equal to half the area of the curved surface ?

15. A right circular cylinder whose length is 1 ft. and the radius of whose base is 6 in. is cut into two segments by a plane parallel to the axis and distant $3\sqrt{3}$ in. from it : find the area of the whole surface of the smaller segment. ($\pi = 3{\cdot}1416$.)

16. If the plane in Example 15 is distant $3\sqrt{2}$ in. from the axis, find the area of the whole surface of the smaller segment. ($\pi = 3{\cdot}1416$.)

17. Verify the statement that to make a vessel of a given capacity, *without a lid*, in the form of a right circular cylinder from a minimum of material, the most advantageous shape is that in which the height is equal to the radius of the base.

18. Verify the statement that to make a vessel of a given capacity, *with a lid*, in the form of a right circular cylinder from a minimum of material, the most advantageous shape is that in which the height is equal to twice the radius of the base.

19. The length of a cylindrical ring is 36 in., and the radius of the cross-section is $1\frac{3}{4}$ in. : find the area of the surface of the ring.

20. The radius of the inner circumference of a cylindrical ring is 7 in., and the diameter of the cross-section is 3 in. : find the area of the surface of the ring.

21. The diameters of the outer and inner circumferences of a cylindrical ring are 9 in. and 7 in. respectively : find the area of the surface of the ring.

22. The area of the surface of a cylindrical ring is $214\frac{1}{2}$ sq. in., and the diameter of the cross-section is $1\frac{1}{2}$ in. : find the radius of the inner circumference.

Examination Questions—XXXI.

($\pi = \frac{22}{7}$.)

Cylinders.

A. *Punjab University: First Exam. in Civil Engineering.*

1. A well is to be dug 5 ft. diameter clear inside, and 36 ft. in depth (excluding the curb), with a brick lining of 9 in. thickness: find plastering in square feet of exposed surface.

B. *Calcutta University: F.E. Exam.*

2. The whole surface of a cylindrical tube is 264 sq. in.: if its length is 5 in. and its external radius 4 in., find its thickness.

C. *Roorkee Engineer: Entrance.*

3. The same number expresses the solidity and convex surface of a cylinder: what is its diameter?

D. *Roorkee Upper Subordinate: Monthly.*

4. What is the proportion between the height of a cylinder and the diameter of its base when the curved surface is equal in area to the two ends?

Rings.

A. *Bombay University: L.C.E. Second Exam.*

5. Investigate the area of the surface of a cylindrical ring.

B. *Roorkee Engineer: Final.*

6. The radius of the inner boundary of a ring is 14 in.; the area of the surface of the ring is 100 sq. in.: find the radius of the outer boundary.

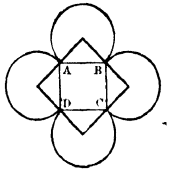
C. *Superior Accounts.*

7. The area of the surface of a ring is 120 sq. in.; the radius of the cross-section is 1 in.: find the length of the ring.

Additional Examination Questions—XXXI.

8. A sheet of metal $\frac{1}{8}$ in. thick is made into a pipe whose internal diameter is half an inch. This pipe is placed round a cylinder 1 ft. in radius: find the area of the external surface of the pipe. (Calcutta University: F.E. Exam.)

9. The cross-section of a pillar 30 ft. high is as shown, a side of the inner square being 2 ft., and the circular segments touching each other at A, B, C, and D: find to the nearest rupee the cost of polishing the exposed surface of the pillar at R. 1 per square foot. (The ends of the pillar are not exposed. (Roorkee Engineer: Final.)

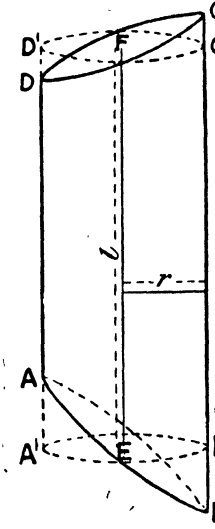


CHAPTER XXXII.

ON OBLIQUE FRUSTA OF RIGHT CIRCULAR CYLINDERS.

PROPOSITION XLIV.

188. *To find the area of the curved surface of an oblique frustum of a right circular cylinder, having given the length of the frustum and the radius of its cross-section.*



Let $ABCD$ be an oblique frustum of a right circular cylinder.

Let its length EF and the radius of its cross-section measure l and r of the same linear unit respectively.

It is required to find the area of the curved surface of $ABCD$ in terms of l and r.

Consider the right circular cylinder $A'B'C'D'$, whose ends lie in parallel planes through E and F.

Since the wedge-shaped portions FCC' and FDD' are equal in all respects, and since the wedge-shaped portions EAA' and EBB' are equal in all respects—

$\therefore$ the curved surface of the cylinder $A'B'C'D'$ is equal in area to the curved surface of the frustum $ABCD$

But the curved surface of the cylinder $A'B'C'D' = 2\pi r \times l$ square units. § 183.

$\therefore$ the curved surface of the frustum $ABCD = 2\pi r \times l$ square units

Hence rule—

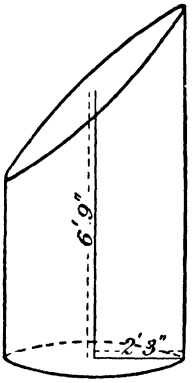
Multiply the number of any linear unit in the length of an oblique frustum of a right circular cylinder by the number of the same linear unit in the circumference of its cross-section, then the product will give the number of the corresponding square unit in the area of its curved surface.

Or briefly—

Curved surface of oblique frustum of right circular cylinder $\Big\}$ = { **circumference of cross-section × length**

$$S = 2\pi rl$$

ILLUSTRATIVE EXAMPLES.



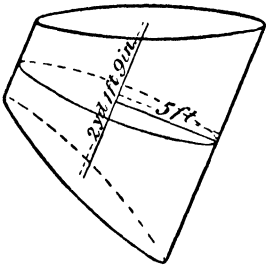
189. *Example* 1.—The radius of the base of a right circular cylinder is 2 ft. 3 in. : find the area of the curved surface of an oblique frustum of this cylinder, if the length of the frustum is 6 ft. 9 in.

Area of curved surface = $2\pi rl$ sq. in. § 188

where $r = 27$,
$l = 81$;

$\therefore$ area of curved surface = $2 \cdot \frac{22}{7} \cdot 27 \cdot 81$ sq. in.
= $13{,}746\frac{6}{7}$ sq. in.
= 10 sq. yd. 5 sq. ft. $66\frac{6}{7}$ sq. in.

Example 2.—The cross-section of an oblique cylinder is a circle whose radius is 5 ft. : find the area of the curved surface of an oblique frustum of this cylinder, if the length of the frustum is 2 yds. 1 ft. 9 in.



Area of curved surface = $2\pi rl$ sq. ft. § 188.

where $r = 5$,
$l = 7\frac{3}{4}$;

$\therefore$ area of curved surface = $10 \times \frac{22}{7} \times \frac{31}{4}$ sq. ft.
= $243\frac{4}{7}$ sq. ft.

Examples—XXXII.

(Take $\pi = \frac{22}{7}$.)

1. The radius of the base of a right circular cylinder is 1 ft. 9 in. : find the area of the curved surface of an oblique frustum of this cylinder, if the length of the frustum is 4 ft. 10 in.

2. The radius of the base of a right circular cylinder is 2 ft. 3 in. : find the area of the curved surface of an oblique frustum of this cylinder, if the length of the frustum is 5 ft. 6 in.

3. The radius of the base of a right circular cylinder is 7 in. : find the area of the curved surface of an oblique frustum of this cylinder, if the length of the frustum is 2 ft. 5 in.

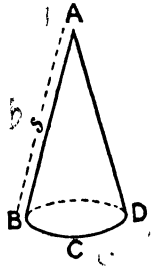
4. The area of the curved surface of an oblique frustum of a right circular cylinder is 7 sq. ft. 48 sq. in. : if the length of the frustum is 2 ft. 4 in., find the radius of the base of the cylinder.

CHAPTER XXXIII.

ON RIGHT CIRCULAR CONES.

PROPOSITION XLV.

190. *To find the area of the curved surface of a right circular cone, having given the slant height of the cone and the circumference of its base.*

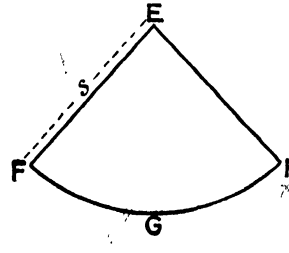


Let $ABCD$ be a right circular cone.

Let its slant height and the circumference of its base measure s and C of the same linear unit respectively.

It is required to find the area of the curved surface of $ABCD$ in terms of s and C.

If we suppose the cone $ABCD$ to be hollow so that it can be cut along the line AB and opened out, it is evident that its curved surface will assume the form of the sector of a circle $EFGH$ whose radius EF is equal to the slant height of the cone (s linear units), and whose arc FGH is equal to the circumference of the base of the cone (C linear units).



$\therefore$ Area of curved surface of cone $=$ area of sector $EFGH$
$= \frac{1}{2} C \cdot s$ square units . § 86.

Hence rule—

Multiply the number of any linear unit in the circumference of the base of a right circular cone by the number of the same linear unit in the slant height, then half the product will give the number of the corresponding square unit in the curved surface.

Or briefly—

Curved surface of right circular cone $=$ **$\frac{1}{2}$(circumference of base) $\times$ (slant height)**

$$S = \tfrac{1}{2}Cs$$

191. This formula can easily be shown to be equivalent to the formula—

$$S = \pi r\sqrt{h^2 + r^2}$$

where h linear units is the height of the cone, and r linear units is the radius of the base of the cone.

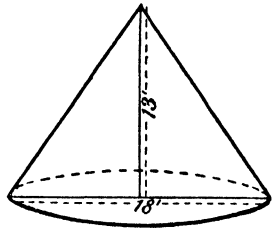
And the *whole* surface of a right circular cone is determined by the expression—

$$\pi r(\sqrt{h^2 + r^2} + r)$$

where h and r have the same significance.

ILLUSTRATIVE EXAMPLES.

192. *Example* 1.—How many yards of canvas, 27 in. wide, are required to construct a conical tent 18 ft. in diameter and 13 ft. in height? ($\pi = 3{\cdot}1416$.)



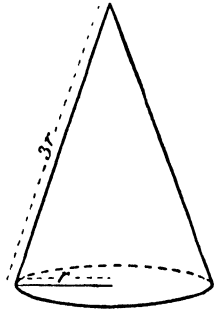
Curved surface of tent $= \pi r\sqrt{h^2+r^2}$ sq. ft. § 191.

where $r = 9$,
$h = 13$;

$\therefore$ curved surface of tent $= 9\pi \cdot \sqrt{13^2 + 9^2}$ sq. ft.
$= 9\pi \cdot \sqrt{250}$ sq. ft.

and length of canvas required $= \dfrac{9\pi \times \sqrt{250} \times 144}{27 \times 36}$ yds. . § 8.
$= 66{\cdot}229$ yds.

Example 2.—The area of the whole surface of a right circular cone is 15 sq. ft., and the slant height is three times the radius of the base: find the radius of the base. ($\pi = 3{\cdot}1416$.)



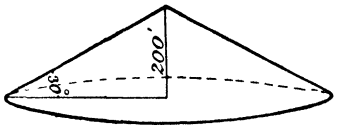
Area of the whole surface of the cone $= \pi r(\sqrt{h^2 + r^2} + r)$ sq. ft. . . § 191.

where $\sqrt{h^2 + r^2} = s = 3r$ . . . . . § 16.

$$\therefore 4\pi r^2 = 15$$
$$r^2 = \frac{15}{4\pi}$$
$$= 1{\cdot}1936\ldots$$
$$r = 1{\cdot}09\ldots$$

radius of base $= 1{\cdot}09$ ft. nearly

Example 3.—A cone is 200 ft. high, and its slant height is inclined 30° to the horizon: find the area of its curved surface in acres. ($\pi = 3{\cdot}1416$.)



If s ft. and r ft. be the slant height and the radius of the base of the cone respectively, we have—

$$s = 400 \quad \text{§ 17.}$$

$$r = \frac{400\sqrt{3}}{2} = 200\sqrt{3} \quad \text{§ 17.}$$

$\therefore$ area of curved surface of cone $= \frac{1}{2} . C . s$ sq. ft. . . § 190.

where $C = 2\pi . 200\sqrt{3}$ . . . . . § 69.

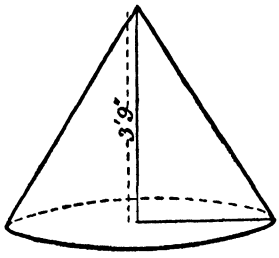
$s = 400$;

hence area of curved surface of cone $= \frac{1}{2} . 2\pi . 200\sqrt{3} . 400$ sq. ft.

$= 435{,}312{\cdot}85$ sq. ft.

$= 9{\cdot}993$ acres

Example 4.—The curved surface of a right circular cone is 32 sq. ft. 56 sq. in., and the height is 3 ft. 9 in. : find the radius of the base.



$$\left.\begin{matrix}\text{Curved surface} \\ \text{of cone}\end{matrix}\right\} = \pi r \sqrt{h^2 + r^2} \text{ sq. in. § 191.}$$

where $h = 45$;

$$\therefore \pi r . \sqrt{45^2 + r^2} = 4664$$

$$r\sqrt{45^2 + r^2} = 1484$$

$$r^2(2025 + r^2) = (1484)^2$$

$$r^2 = 784$$

$$\therefore r = 28$$

radius of base = 2 ft. 4 in.

Examples—XXXIII.

(Take $\pi = \frac{22}{7}$.)

Find the areas of the *curved* surfaces of the following right circular cones, having—

1. Circumference of base 3 ft. 6 in. ; slant height 1 ft. 10 in.
2. Circumference of base 4 ft. 7 in. ; slant height 2 ft. 5 in.
3. Diameter of base 2 ft. 3 in. ; slant height 1 ft. 10 in.
4. Radius of base 1 ft. 2 in. ; slant height 2 ft.
5. Radius of base 8 in. ; height 15 in.
6. Diameter of base 1 ft. 10 in. ; height 5 ft.
7. Diameter of base 1 ft. 6 in. ; height 3 ft. 4 in.
8. Circumference of base 3 ft. 8 in. ; height 2 ft.

Find the areas of the *whole* surfaces of the following right circular cones, having—

9. Radius of base 2 ft. 3 in. ; slant height 4 ft.
10. Radius of base 3 ft. 6 in. ; slant height 5 ft. 3 in.
11. Circumference of base 7 ft. 4 in. ; slant height 2 ft. 6 in.
12. Radius of base 5 in. ; height 1 ft.
13. Diameter of base 2 ft. 6 in. ; height 9 ft 4 in.
14. Circumference of base 11 ft. ; height 6 yds. 4 in.
15. The curved surface of a right circular cone is 176 sq. in., and the slant height is 8 in. : find the radius of the base.

16. The curved surface of a right circular cone is 66 sq. in., and the radius of the base is 3·5 in. : find the slant height.

17. The curved surface of a right circular cone is 47½ sq. in., and the radius of the base is 3 in. : find the slant height.

18. The curved surface of a right circular cone is 204¾ sq. in., and the height is 12 in. : find the radius of the base.

19. The curved surface of a right circular cone is 550 sq. in., and the height is 2 ft. : find the radius of the base.

20. The curved surface of a right circular cone is 14,586 sq. in., and the height is 220 in. : find the slant height.

Examination Questions—XXXIII.

($\pi = \frac{22}{7}$.)

A. *Punjab University : First Exam. in Civil Engineering.*

1. How much canvas will make a conical tent 11 ft. in height, and 12 ft. in diameter at the base ?

B. *Sibpur Engineer : Annual Exam.*

2. A right-angled triangle, of which the sides are 3 in. and 4 in. in length is made to turn round on the longer side : find the area of the whole surface of the cone thus formed.

C. *Roorkee Engineer : Entrance.*

3. Calculate to three places of decimals the entire superficial area in square inches of a solid cone, the diameter of its base being 8 in., and its altitude 13 in.

4. The area of the whole surface of a right circular cone is 32 sq. ft., and the slant height is three times the radius of the base : find the volume of the cone.

D. *Roorkee Upper Subordinate : Entrance.*

5. Find the cost of painting a conical spire 64 ft. in circumference at the base, and 108 ft. in slant height, at 7½*d.* per square yard.

E. *Staff College.*

6. Find the cost of the canvas for 150 conical tents, the height of each being 11¼ ft., and the diameter of the base 12 ft., at 5*d.* per square yard.

Additional Examination Questions—XXXIII.

7. Determine an expression for the curved surface of a right circular cone. (Roorkee Engineer : Entrance.)

8. Find what length of canvas ¾ yd. wide is required to make a conical tent 12 ft. in diameter and 8 ft. high. (Punjab University : First Exam. in Civil Engineering.)

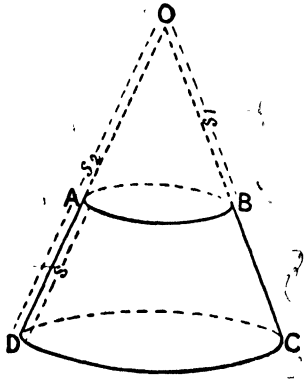
9. How many square yards of canvas will be required for a tent the walls of which form a right circular cylinder 10 ft. in diameter and 8 ft. high, the roof of the tent being a right circular cone with the apex 12 ft. above the ground? The roof does not extend beyond the top of the wall. (Roorkee Engineer : Final.)

CHAPTER XXXIV.

ON FRUSTA OF RIGHT CIRCULAR CONES.

PROPOSITION XLVI.

193. *To find the area of the curved surface of a frustum of a right circular cone, having given the circumferences of its ends and its slant thickness.*



Let $ABCD$ be a frustum of the right circular cone ODC.

Let the circumferences of its larger and smaller ends measure C and c of the same linear unit respectively.

Let its slant thickness measure s of the same linear unit.

It is required to find the area of the curved surface of $ABCD$ in terms of C, c, and s.

Let OB and OD measure s_1 and s_2 of the same linear unit respectively.

Now—

$$\left.\begin{array}{r}\text{Curved surface of}\\ \text{frustum } ABCD\end{array}\right\} = \left\{\begin{array}{l}\text{curved surface of cone } OCD - \text{curved}\\ \text{surface of cone } OAB\end{array}\right.$$

$$= (\tfrac{1}{2} \cdot C \cdot s_2 - \tfrac{1}{2} \cdot c \cdot s_1) \text{ sq. units} \quad . \quad \text{§ 190.}$$

$$= \{\tfrac{1}{2} \cdot C \cdot (s_1 + s) - \tfrac{1}{2} \cdot c \cdot (s_2 - s)\} \text{ sq. units}$$

$$= \{\tfrac{1}{2} \cdot (C + c)s + \tfrac{1}{2}(Cs_1 - cs_2)\} \text{ sq. units}$$

But $c : C = s_1 : s_2$ . . . . . § 177.

$\therefore Cs_1 = cs_2$ . . . . . Euc. VI. 16.

$\therefore Cs_1 - cs_2 = 0$

$$\left.\begin{array}{r}\therefore \text{curved surface of}\\ \text{frustum } ABCD\end{array}\right\} = \tfrac{1}{2}(C + c)s \text{ sq. units}$$

Hence rule—

Multiply the number of any linear unit in the slant thickness of a frustum of a right circular cone by the number of the same linear unit in the sum of the circumferences of the two ends, then half the

product will give the number of the corresponding square unit in the curved surface.

Or briefly—

$$\left.\begin{array}{l}\textbf{Curved surface of}\\ \textbf{frustum of right}\\ \textbf{circular cone}\end{array}\right\} = \left\{\begin{array}{l}\tfrac{1}{2}\text{(slant thickness) } \times \text{ (sum}\\ \text{of circumferences of ends)}\end{array}\right.$$

$$\mathbf{S = \tfrac{1}{2} . s . (C + c)}$$

194. This formula can easily be shown to be equivalent to the formula—

$$S = \pi(R + r)s$$

where R linear units and r linear units are the radii of the two ends of the frustum, and s linear units is the slant thickness of the frustum.

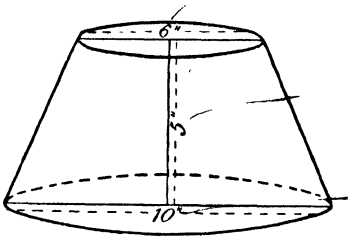
And the *whole* surface of a frustum of a right circular cone is determined by the expression—

$$\pi(R^2 + r^2 + Rs + rs)$$

where R, r, and s have the same significance.

ILLUSTRATIVE EXAMPLES.

195. *Example* 1.—A frustum of a right circular cone has a diameter of base 10 in., of top 6 in., and a height of 5 in.: find the area of its whole surface. ($\pi = 3{\cdot}1416$.)



If s in. = slant height of frustum, we have—

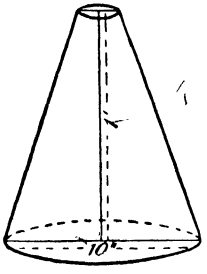
$$s = \sqrt{5^2 + 2^2} \quad . \quad . \quad \text{§ 16.}$$
$$= \sqrt{29}$$
$$= 5{\cdot}3851$$

$\therefore$ whole surface of frustum $= \pi(R^2 + r^2 + Rs + rs)$ sq. in. . § 194.

where $R = 5$,
$r = 3$,
$s = 5{\cdot}3851$;
hence—

whole surface of frustum $= \pi(25 + 9 + 5{\cdot}3851 \times 8)$ sq. in.
$= 242{\cdot}15$ sq. in.

Example 2.—A frustum of a right circular cone is 1 ft. high, with a diameter at the base of 10 in. If the top of the frustum has an area



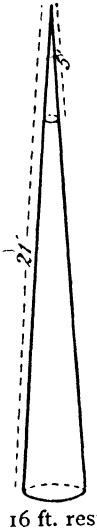
of π sq. in., find the curved surface of frustum. ($\pi = 3\ 1416$.)

Radius of top of frustum $= \sqrt{\frac{\pi}{\pi}}$ in. . . . § 71.
$= 1$ in.

$\therefore$ curved surface of frustum $= \pi(R+r)s$ sq. in. § 194.

where $R = 5$,
$r = 1$,
$s = \sqrt{12^2 + 4^2} = \sqrt{160}$ . . . . § 16.

hence curved surface of frustum $= \pi \times 6 \times \sqrt{160}$ sq. in.
$= 238{\cdot}42$ sq. in.



Example 3.—If from a right circular cone whose slant height is 21 ft., and circumference of base 8 ft., there be cut off by a plane parallel to its base a cone of slant height 5 ft., find the curved surface of the remaining frustum.

If c ft. = circumference of top of frustum

we have, by similar figures—

$c : 8 = 5 : 21$ . . . . . § 177.
$c = \frac{40}{21}$

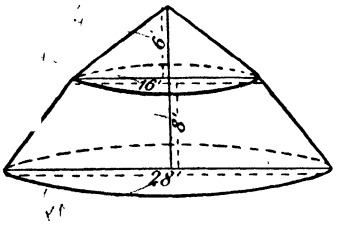
$\therefore$ area of curved surface of frustum $= \frac{1}{2}s(C + c)$ sq. ft. . . § 193.

where $C = 8$,
$c = \frac{40}{21}$,
$s = 16$;
hence—

area of curved surface of frustum $= \frac{1}{2} . 16 . (8 + \frac{40}{21})$ sq. ft.
$= 79\frac{5}{21}$ sq. ft.

Example 4.—A tent is made in the form of a frustum of a right circular cone surmounted by a cone: find the number of square yards of canvas required for the tent, supposing the diameters of the ends of the frustum to be 28 ft. and 16 ft. respectively, the height of the frustum 8 ft., and the height of the conical part 6 ft. ($\pi = 3{\cdot}1416$.)

Curved surface of cone $= \frac{1}{2} . \times \pi . 16 \times \sqrt{6^2 + 8^2}$ sq. ft. § 190.
$= 80\pi$ sq. ft.



Curved surface of frustum $= \pi(R + r)s$ sq. ft. § 194.

where $R = 14$,
$r = 8$,
$s = \sqrt{8^2 + 6^2} = 10$. § 16.

hence—

Curved surface of frustum $= \pi(14+8)10$ sq. ft.
$= 220\pi$ sq. ft.

$\therefore$ square yards of canvas required $= \frac{\pi \times (80 + 220)}{9}$
$= 104{\cdot}72$

Examples—XXXIV.

(Take $\pi = \frac{22}{7}$ unless otherwise stated.)

Find the areas of the curved surfaces of the following frusta of right circular cones, having—

1. Circumferences of ends 13 in. and 16 in. ; slant thickness 6 in.

2. Circumferences of ends 3 ft. 4 in. and 4 ft. 8 in. ; slant thickness 1 ft. 6 in.

3. Radii of ends 14 in. and 21 in. ; slant thickness 8 in.

4. Radii of ends 5 ft. 3 in. and 6 ft. 5 in. ; slant thickness 2 ft. 4 in.

Find the areas of the whole surfaces of the following frusta of right circular cones, having—

5. Radii of ends 7 in. and 14 in. ; slant thickness 10 in.

6. Radii of ends 2 ft. 11 in. and 3 ft. 6 in. ; slant thickness 2 ft. 4 in.

7. Circumferences of ends 11 ft. and 14 ft. 8 in. ; slant thickness 10 in.

8. Circumferences of ends 14 ft. 8 in. and 18 ft. 4 in. ; slant thickness 1 ft.

9. Find the curved surface of a frustum of a right circular cone whose thickness is 4 in., and the radii of whose ends are 7 in. and 10 in.

10. Find the area of the curved surface of a frustum of a right circular cone whose thickness is 1 ft., and the radii of whose ends are 10 in. and 15 in.

11. Find the cost at *2s. 6d.* per square foot of enamelling the inside of an open vessel in the form of a frustum of a cone, if the depth of the vessel is 5 ft., and if the diameters of the ends are 5 ft. 2 in. and 3 ft. 4 in. respectively.

12. The height of a frustum of a right circular cone is 12 in. : find the area of the curved surface if the cone from which the frustum has been obtained had the following dimensions: Height 32 in., diameter of base 16 in. ($\pi = 3{\cdot}1416$.)

Examination Questions—XXXIV.

(Take $\pi = \frac{22}{7}$, unless otherwise stated.)

A. *Bombay University, Diploma of Agriculture : Second Exam.*

1. A tin funnel consists of two parts : one part is conical, the slant side is 6 in., the circumference of one end is 20 in., and of the other end $1\frac{1}{2}$ in. ; the other part is cylindrical, the circumference being $1\frac{1}{2}$ in., and the length 8 in. Find the number of square inches of tin.

B. *Bombay University : L.C.E. Second Exam.*

2. The half of a regular hexagon, formed by joining the middle points of two opposite sides of the whole figure, revolves about this line : determine the whole surface of the solid thus generated, a side of the hexagon being 10 ft.

C. *Punjab University : First Exam. in Civil Engineering.*

3. Give the formulæ for finding the surface of a frustum of a cone.

4. What is the area of the slant surface of a frustum of a right cone, the area of the two circular ends being 1256·64 in. and 78·54 in. respectively, and the vertical height of the frustum 20 in. ? ($\pi = 3{\cdot}1416$)

D. *Madras University: B.E. Exam.*

5. Prove that the area of the curved surface of a frustum of a right circular cone is obtained by multiplying the circumference of the mean section by the slant height.

E. *Roorkee Engineer: Entrance.*

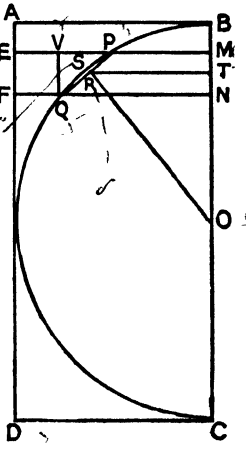
6. A tent is made in the form of a conic frustum, surmounted by a cone. The diameters of the base and the top of the frustum are 14 and 7 ft., its height 8 ft., and the height of the tent 12 ft. : find the quantity of canvas required for it.

CHAPTER XXXV.

ON SPHERES, SEGMENTS OF SPHERES, AND ZONES OF SPHERES.

PROPOSITION XLVII.

196. *The curved surface of a sphere, or of a segment of a sphere, or of a zone of a sphere is equal in area to the curved surface of the corresponding zone of the cylinder that circumscribes the sphere.*



Consider the rectangle $ABCD$ circumscribing the semicircle $BPQC$.

Let EM, FN be two straight lines parallel to AB, meeting the semicircle in P and Q. If the whole figure is made to revolve about BC, the semicircle $BPQC$ will sweep out a sphere, the straight line AD will sweep out its circumscribing cylinder, the arc PSQ will sweep out a zone of the sphere, the straight line PQ will sweep out the corresponding frustum of a cone, and the straight line EF the corresponding zone of the circumscribing cylinder.

It is required to show that the spherical zone swept out by the arc PSQ is equal in the area of its curved surface to the corresponding zone of the circumscribing cylinder swept out by the straight line EF.

Join RT, the middle points of PQ and MN.

Join R to O, the centre of the sphere.

Draw QV perpendicular to EM.

Because the triangles ORT, PQV are similar.

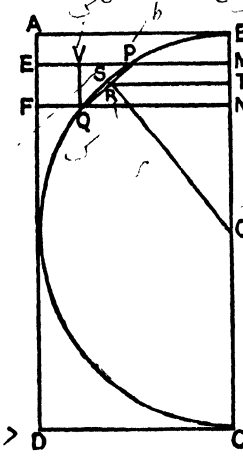
$$\therefore OR : RT = PQ : QV \quad \ldots\ldots\ldots \S\ 66.$$

$$\therefore OR \times QV = RT \times PQ \quad \ldots\ldots \text{Euc. VI. 16.}$$

$$\therefore 2\pi \,.\, OR \times QV = 2\pi \,.\, RT \times PQ$$

But $2\pi \cdot RT \times PQ = \pi \cdot (PM + QN) \times PQ$
= area of curved surface of frustum of cone swept out by the straight line PQ § 194.
$\therefore 2\pi \cdot OR \times QV$ = area of curved surface of frustum of cone swept out by the straight line PQ

Now, by taking the point P nearer and nearer to the point Q, the straight line PQ is made to differ less and less from the arc PSQ, and the straight line OR from the radius of the sphere.



Therefore, in the limit, when OR is equal to the radius of the sphere, the curved surface of the conical frustum swept out by the straight line PQ is equal to the curved surface of the spherical zone swept out by the arc PSQ.

That is, in the limit, $2\pi \cdot OR \times QV =$ area of curved surface of spherical zone swept out by the arc PSQ.

And, in the limit, $OR = OB$.

But $2\pi \cdot OB \times QV =$ area of curved surface of zone of circumscribing cylinder swept out by the straight line EF . § 183.

Therefore, in the limit, area of curved surface of zone of circumscribing cylinder swept out by the straight line EF = area of curved surface of spherical zone swept out by the arc PSQ.

But the surface of the whole sphere, or of any segment of the sphere, or of any zone of the sphere, is the sum of all such elementary zones.

Therefore, the curved surface of a sphere, or of a segment of a sphere, or of a zone of a sphere, is equal in area to the curved surface of the corresponding zone of the cylinder that circumscribes the sphere.

From this result it is easy to deduce rules for determining—

A. The surface of a sphere.

B. The curved surface of a segment of a sphere or of a zone of a sphere.

This we will proceed to do.

197. A. The surface of a sphere.

Let the diameter of the sphere measure d of any linear unit.

Now, since the surface of a sphere is equal in area to the curved surface of its circumscribing cylinder . . . . . § 196.

And since the diameter of the base of this circumscribing cylinder and its height each measure d of the same linear unit.

$\therefore$ surface of sphere $= \pi d \times d$ sq. units . . . . . § 183.
$= \pi d^2$ sq. units

Hence rule—

Multiply the square of the number of any linear unit in the diameter of a sphere by π, then the product will give the number of the corresponding square unit in the surface.

Or briefly—

Surface of sphere $= \pi$(diameter)^2
$S = \pi d^2$

198. B. **The curved surface of a segment of a sphere or of a zone of a sphere.**

Let the diameter of the sphere measure d of any linear unit.

Let the height of the segment (or of the zone) measure h of the same linear unit.

Now, since the curved surface of a segment of a sphere (or of a zone of a sphere) is equal in area to the curved surface of the corresponding zone of the circumscribing cylinder . . . . § 196

And since the height of this zone of the circumscribing cylinder measures h, and the diameter of its base measures d of the same linear unit.

$\therefore$ curved surface of segment (or zone) $= \pi d \times h$ sq. units . § 183.

Hence rule—

Multiply the number of any linear unit in the height of a segment of a sphere (or of a zone of a sphere) by the number of the same linear unit in the diameter of the sphere, then π times the product will give the number of the corresponding square unit in the curved surface of the segment (or of the zone).

Or briefly—

Curved surface of segment of sphere (or of zone of sphere) = π × diameter of sphere × height of segment (or of zone)
$S = \pi dh$

ILLUSTRATIVE EXAMPLES.

Spheres.

199. *Example* 1.—What would be the expense of gilding a spherical ball of 6 ft. diameter at $3\frac{1}{2}d.$ the square inch?

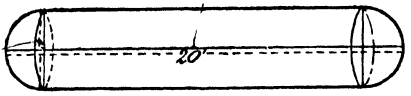
Surface of ball $= \pi d^2$ sq. in. . . . . § 197.

where $d = 72$;

$\therefore$ surface of ball $= \pi . (72)^2$ sq. in.

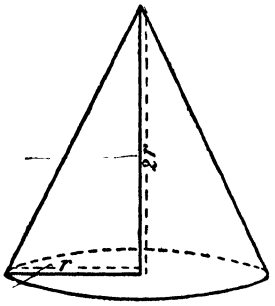
Hence—

Expense of gilding at $3\frac{1}{2}d.$ the sq. inch $= \frac{22}{7} \times 72 \times 72 \times \frac{7}{2}d.$
$= £237$ 12*s.* 0*d.* nearly



Example 2.—Find the cost of painting a pontoon with hemispherical ends at 2*s.* 6*d.* per square foot. The length of the pontoon is 20 ft., and its diameter is 4 ft.

Surface of pontoon = curved surface of a cylindrical portion + curved surface of two equal hemispherical portions
$= (\pi \times 4 \times 16 + \pi \times 4^2)$ sq. ft. . . §§ 183, 197.
∴ cost of painting at 2*s.* 6*d.* per sq. foot. $= \pi(64 + 16) \times 30d.$
$= \frac{22}{7} \times 80 \times 30d.$
$= 7542\frac{6}{7}d.$ nearly
$= £31\ 8s.\ 7d.$ nearly



Example 3.—A sphere has the same volume as a right circular cone with its height equal to twice the radius of its base: express the curved surface of the cone as a decimal fraction of the surface of the sphere.

If the diameter of the sphere and the radius of the base of the cone measure d and r of the same linear unit respectively, we have—

$$\frac{\pi d^3}{6} = \frac{1}{3} \cdot \pi r^2 \times 2r \quad \text{§§ 166, 142.}$$
$$d^3 = 4r^3$$
$$d^2 = \sqrt[3]{16} \times r^2$$

Now, curved surface of cone $= \pi r \sqrt{h^2 + r^2}$ sq. units . § 191.
where $h = 2r$;

∴ curved surface of cone $= \pi r^2 \sqrt{5}$ sq. units
and surface of sphere $= \pi d^2$ sq. units . . . § 197.
$= \pi \sqrt[3]{16} \times r^2$ sq. units

$$\therefore \frac{\text{curved surface of cone}}{\text{surface of sphere}} = \frac{\pi \cdot r^2 \cdot \sqrt{5}}{\pi \sqrt[3]{16} \cdot r^2}$$
$$= \frac{\sqrt{5}}{\sqrt[3]{16}}$$
$$= 0{\cdot}887 \ldots$$

Example 4.—A sphere and a cube have the same volume: show that the surface of the cube is 1·2407 times that of the sphere. ($\pi = 3{\cdot}14159$.)

Let the edge of the cube and the diameter of the sphere measure a and d of the same linear unit respectively.

Then—

$$a^3 = d^3 \times \frac{\pi}{6} \quad \ldots \ldots \ldots \quad \text{§§ 117, 166.}$$
$$\therefore a = d \times \sqrt[3]{\frac{\pi}{6}}$$

$$\therefore 6a^2 = 6 \times d^2 \times \sqrt[3]{\left(\frac{\pi}{6}\right)^2}$$

$$= \pi \times d^2 \times \frac{6}{\pi}\sqrt[3]{\left(\frac{\pi}{6}\right)^2}$$

$$= \pi d^2 \times \sqrt[3]{\left(\frac{6}{\pi}\right)}$$

That is—

$$\text{surface of cube} = \text{surface of sphere} \times \sqrt[3]{\left(\frac{6}{\pi}\right)}$$

$$\text{But } \frac{6}{\pi} = 1{\cdot}909860$$

and $\sqrt[3]{1{\cdot}909860}$ will be found to be 1·2407

Therefore, what was required has been done.

Examples—XXXV. A.

(Take $\pi = \frac{22}{7}$.)

Spheres.

Find the areas of the surfaces of the following spheres, having—

1. Radius 7 in.
2. Radius 2 ft. 4 in.
3. Diameter 3 ft. 6 in.
4. Diameter 1 yd. 2 ft. 9 in.
5. Circumference 3 ft. 8 in.
6. Circumference 9 ft. 2 in.
7. Volume 22 cub. ft. 792 cub. in.
8. Volume 1 cub. ft.

Find the radii of the following spheres, having—

9. Surface 616 sq. in.
10. Surface 264 sq. in.
11. Surface 17 sq. ft. 16 sq. in.

12. A solid is composed of a cylinder with hemispherical ends. If the cylindrical portion is 6 ft. long and 4 ft. in diameter, find the area of the whole surface.

13. Find the cost of painting a hemispherical dome 46 ft. in diameter, at the rate of 3*d.* per square yard.

14. A solid is composed of a cylinder with hemispherical ends. If the whole length of the solid is 9 ft., and if its diameter is 3 ft., find the cost of polishing its surface at the rate of 1*s.* 6*d.* per square foot.

15. Find, to the hundredth part of an inch, the radius of a sphere whose surface is equal to the whole surface of a right circular cone whose height is 6 in., and the diameter of whose base is 4 in.

16. Find the internal surface of a spherical shell of thickness 1 in., and of external radius 5·5 in.

17. Find the cost of varnishing the whole surface of a hemispherical bowl, $\frac{1}{2}$ in. thick and 8 in. in internal diameter, at the rate of 3*d.* per square inch.

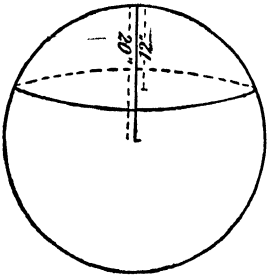
18. A solid is composed of a right circular cone and hemisphere having the same circular base: find the whole surface of the solid if the height of the cone is 2 ft., and the diameter of the common circular base 1 ft.

19. Find the ratio of the surface of a cube to the surface of its inscribed sphere.

20. The vertical angle of a right circular cone is 90°, and its height is 10 in.: find the surface of the greatest inscribed sphere.

ILLUSTRATIVE EXAMPLES.

Segments and Zones of Spheres.

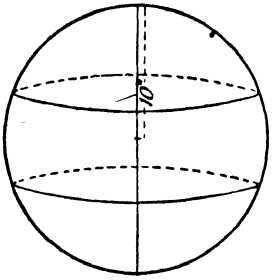


200. *Example* 1.—Find the area of the curved surface of a segment of a sphere. Height of segment 12 in., radius of sphere 20 in. ($\pi = 3{\cdot}1416$.)

$$\left.\begin{matrix}\text{Curved surface of}\\ \text{segment}\end{matrix}\right\} = \pi dh \text{ sq. in.} \quad \S\ 198.$$

where $d = 40$,
$h = 12$;

$$\therefore \text{ curved surface of segment} = \pi \times 40 \times 12 \text{ sq. in.}$$
$$= 1507{\cdot}968 \text{ sq. in.}$$

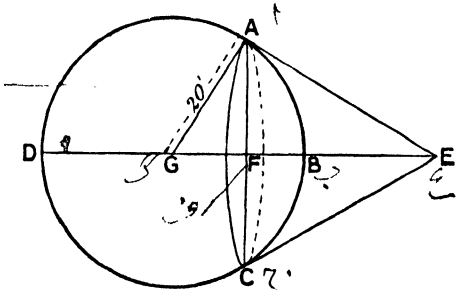


Example 2.—Divide the surface of a sphere of 10 in. radius into three equal parts by parallel sections, and find the heights of the two segments and of the middle zone.

Since the curved surface of each part is one-third the surface of the sphere, therefore the height of each part must be one-third the diameter of the sphere

$$= 6\tfrac{2}{3} \text{ in.}$$

Example 3.—Find the distance from a sphere of radius 20 ft. from which one-fourth of its surface can be seen.



From the point E, outside the sphere $ABCD$, the curved surface of the segment ABC is visible.

If, then, one-fourth part of the surface of the sphere is visible from E, we must have—

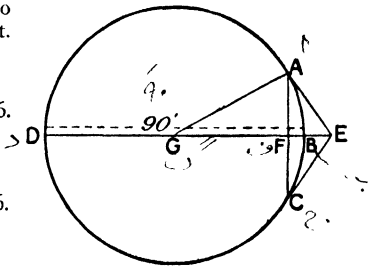
$$BF = \tfrac{1}{4}.BD \quad \S\ 196.$$
$$= 10 \text{ ft.}$$
$$\therefore FG = 10 \text{ ft.}$$

But, by similar figures—

$$EG : AG = AG : FG \quad \ldots\ldots\ldots \quad \S\ 66$$
$$EG : 20 \text{ ft.} = 20 : 10$$

$\therefore EG = 40$ ft.
$\therefore EB = 20$ ft.

Example 4.—A sphere is 90 ft. in diameter: find what fraction of the whole surface will be visible to an eye placed at a distance of 8 ft. from the surface.



Fraction of surface of sphere $ABCD$ visible from E $= \frac{BF}{BD}$ § 196.

But, by similar figures—

FG : 45 ft. = 45 ft. : EG § 66.
and $EG = (8 + 45)$ ft.
$= 53$ ft.

$$\therefore FG = \frac{45 \times 45}{53} \text{ ft.}$$

$$\therefore BF = \left(45 - \frac{45 \times 45}{53}\right) \text{ ft.}$$

$$= \frac{45 \times 8}{53} \text{ ft.}$$

$$\text{hence required fraction} = \frac{45 \times 8}{53 \times 90}$$

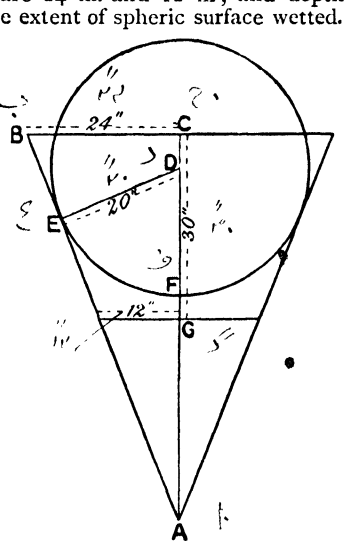
$$= \tfrac{4}{53}$$

Example 5.—A sphere of 20 in. radius is placed inside a frustum of a cone (the radii of whose ends are 24 in. and 12 in., and depth 30 in.), which is full of water: find the extent of spheric surface wetted.

We must first ascertain in what position the sphere will rest.

Let the figure represent a section of the sphere and frustum through the axis of the frustum.

Complete the cone of which the frustum is a part.



By similar figures—

AG : AG + 30 in. = 12 : 24 § 66.
$\therefore AG = 30$ in.

Again, by similar figures—

AD : DE = AB : BC . § 66.
or AD : 20 in. = $\sqrt{BC^2 + AC^2}$: BC . . § 16.
$= \sqrt{24^2 + 60^2} : 24$
$= \sqrt{4176} : 24$
$\therefore AD = \frac{5\sqrt{4176}}{6}$ in.
$= 53{\cdot}8516$ in.
$\therefore DG = (53{\cdot}8516 - 30)$ in.
$= 23{\cdot}8516$ in.
and $FG = 3{\cdot}8516$ in.

which shows that the sphere does not rest on the bottom of the frustum.

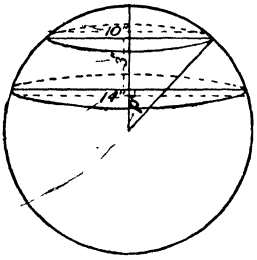
Hence—

Extent of spheric surface wetted = πdh sq. in. . . § 198.

where $d = 40$,
$h = 30 - 3{\cdot}8516 = 26{\cdot}1484$;
that is—

extent of spheric surface wetted = $\frac{22}{7} \times 40 \times 26{\cdot}1484$ sq. in. nearly
= 3284·2 sq. in. nearly

Example 6.—A zone of a sphere is 3 in. in thickness, the diameter of the base is 14 in., and that of the top 10 in.: find the curved surface. ($\pi = 3{\cdot}1416$.)



Let p in. = perpendicular distance of the base of the zone from the centre of the sphere of which it is a zone.

Then, if r in. = radius of this sphere, we have—

$$\left.\begin{aligned}(p+3)^2 + 5^2 &= r^2\\ \text{and } p^2 + 7^2 &= r^2\end{aligned}\right\} \quad \S\ 16.$$

$$\therefore p = \tfrac{5}{2}$$

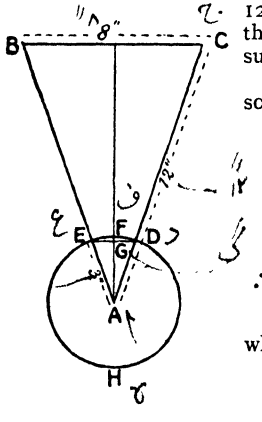
$$\text{and } r = \sqrt{\tfrac{25}{4} + 7^2} = \frac{\sqrt{221}}{2}$$

$$\left.\begin{aligned}\therefore \text{ curved surface of}\\ \text{zone}\end{aligned}\right\} = \pi dh \text{ sq. in. . } \S\ 198.$$

where $d = \sqrt{221}$,
$h = 3$;
hence

curved surface of zone = $\pi \times \sqrt{221} \times 3$ sq. in.
= 140·109 sq. in.

Example 7.—A right cone, base 8 in. diameter, and slant height 12 in., is set into a sphere of radius 3 in., so that vertex and centre coincide: find the surface of the solid. ($\pi = 3{\cdot}1416$.)



Let the figure represent a section of the solid through the axis of the cone.

By similar figures—

$$ED : BC = 3 : 12 \quad \S\ 66.$$
$$\therefore ED = 2 \text{ in.}$$
$$\text{also } AG = \sqrt{9-1} \text{ in. . } \S\ 16.$$
$$= 2\sqrt{2} \text{ in.}$$

$$\left.\begin{aligned}\therefore \text{ curved surface of spherical segment } EHD\end{aligned}\right\} = \pi dh \text{ sq. in. } \S 198.$$

where $d = 6$,
$h = 2\sqrt{2} + 3$;

Hence—

curved surface of spherical segment EHD $= \pi \times 6 \times (2\sqrt{2} + 3)$ sq. in.

also curved surface of conical frustum $BCDE$ $= \frac{1}{2}(\pi \times 8 + \pi \times 2) \cdot 9$ sq. in. § 193.

$= \pi \times 45$ sq. in.

and base of cone ABC $= \pi \times 4^2$ sq. in. . . . . § 71.

$= \pi \times 16$ sq. in.

hence surface of whole solid $= \pi\{6(2\sqrt{2} + 3) + 45 + 16\}$ sq. in.

$= \pi \times 95{\cdot}9705$ sq. in.

$= 301{\cdot}5$ sq. in.

Examples—XXXV. B.

Segments and Zones of Spheres.

(Take $\pi = \frac{22}{7}$, unless otherwise stated.)

Find the areas of the curved surfaces of the following segments of spheres, having—

1. Circumference of sphere 18 in. ; height of segment 5 in.

2. Radius of sphere 7 in. ; height of segment 2·3 in.

3. Diameter of sphere 3·5 in. ; height of segment 0·75 in.

Find the areas of the curved surfaces of the following zones of spheres, having—

4. Circumference of sphere 2 ft. 6 in. ; height of zone 7 in.

5. Radius of sphere 2·8 in. ; height of zone 1·3 in.

6. Diameter of sphere 8·4 in. ; height of zone 3·2 in.

7. Find the whole surface of a segment of a sphere if the height of the segment is 3 in., and if the radius of the sphere is 15 in.

8. Find the whole surface of a segment of a sphere if the height of the segment is 4 ft., and if the circumference of the sphere is 62$\frac{8}{7}$ ft.

9. The diameter of a sphere is 25 ft. : find the whole surface of a zone of this sphere whose plane ends are distant 3 ft. 6 in. and 7 ft. 6 in. respectively from the centre of the sphere and on the same side of it.

10. The diameter of a sphere is 25 ft. : find the whole surface of a zone of this sphere whose plane ends are distant 3 ft. 6 in. and 7 ft. 6 in. respectively from the centre of the sphere and on opposite sides of it.

11. Find the curved surface of a segment of a sphere in which the radius of the base is 6 in. and the height 3 in. ($\pi = 3{\cdot}1416$.)

12. Find the curved surface of a zone of a sphere in which the radii of the plane ends are 4 in. and 5 in. respectively, and the thickness 1 in. ($\pi = 3{\cdot}1416$.)

13. Find at what distance from the surface of a sphere an eye must be placed to see one-tenth of the surface, the radius of the sphere being 1 ft.

14. What fraction of the surface of a sphere will be visible to an eye placed at a distance of 3 ft. from the centre, if the diameter of the sphere is 1 ft. ?

Examination Questions—XXXV.

(Take $\pi = \frac{22}{7}$, unless otherwise stated.)

Spheres.

A. *Bombay University: L.C.E. Second Exam.*

1. Find the expense of painting a cylindrical pontoon with hemispherical ends at 6*d.* per square yard, the length of the cylindrical part being 19 ft. 4 in., and the common diameter of the cylinder and hemispheres being 2 ft. 8 in.

B. *Punjab University: First Exam. in Civil Engineering.*

2. A sphere is 36 in. in diameter : find the area of its surface in square inches. ($\pi = 3{\cdot}14159$.)

3. A circular room has perpendicular walls 15 ft. high, the diameter of the room being 28 ft. ; the roof is a hemispherical dome : find the cost of plastering the whole surface at 9*d.* per square foot.

4. Assuming the earth to be a sphere with a diameter of 5,000,000 ft., find the area of its surface in square miles.

5. What is the surface of a sphere whose diameter is 21 in. ?

6. Find the volume of a sphere when its surface is equal to that of a circle 9 ft. in diameter.

C. *Calcutta University: F.E. Exam.*

7. A cathedral has two spires and a dome ; each of the former consists, in the upper part, of a pyramid 60 ft. high, standing on a square base, of which a side is 20 ft. The dome is a hemisphere of 40 ft. radius. Find the cost of covering the three with lead at $7\frac{1}{2}d.$ per square foot. ($\pi = 3{\cdot}1416$.)

D. *Sibpur Engineer: Annual Exam.*

8. A wrought-iron cylindrical boiler 10 ft. long, 4 ft. in diameter, and $\frac{3}{8}$ in. thick (inside measurements), is closed by hemispherical ends : find the external surface.

9. A cylinder 12 ft. high and 6 ft. in diameter is surmounted by a cone also 6 ft. in diameter and 4 ft. high : find the radius of a hemisphere whose entire surface is equal to the united curved surfaces of the cone and the cylinder.

E. *Roorkee Engineer: Entrance.*

10. The price of a ball at 1*d.* the cubic inch is as great as the cost of gilding it at 3*d.* the square inch. : what is its diameter ?

F. *Roorkee Upper Subordinate: Entrance.*

11. A sphere and a cube have the same surface : show that the volume of the sphere is 1·38 times that of the cube. ($\pi = 3{\cdot}1416$.)

12. A cylinder 24 ft. long and 4 ft. in diameter is closed by a hemisphere at each end : find the area of the whole surface.

G. *Roorkee Engineer: Final.*

13. A sphere has the same number of cubic feet in its volume as it has square feet in its surface : find the diameter.

Segments of Spheres.

A. *Bombay University: L.C.E. Second Exam.*

14. Find at what distance from the surface of a sphere an eye must be placed to see one-sixth of the surface, the diameter of the sphere being 1 ft.

B. *Punjab University: First Exam. in Civil Engineering.*

15. How much of the earth's surface could a person see if he were raised 6 miles above it? (The diameter of the earth = 7912 miles.)

16. A cast-iron shell 12 in. in external diameter floats in water and is immersed 10 in. Find the area of the immersed surface in square feet; multiply it by 62·5, and the result will be the weight of the shell in pounds. Supposing, then, that cast iron weighs 435 lbs. to a cubic foot, what will be the thickness of the shell?

C. *Madras University: B.E. Exam.*

17. Find the whole surface of a segment of a sphere when the radius of the base is 16 ft., and height of segment 5 ft.

18. Find the area of a spherical dome, the diameter of the base being 25 ft. and the height 9 ft.

D. *Calcutta University: F.E. Exam.*

19. A hollow paper cone, whose vertical angle is 60°, is held with its vertex downwards, and in it there is placed a sphere of radius of 2 in. The portion of the cone remote from the apex is now cut away along the line where the paper touches the sphere. Find the exterior surface of the body thus formed.

E. *Sibpur Apprentice Dept.: Annual Exam.*

20. The radius of a sphere is 12 ft.; from a point which is at a distance of 15 ft. from the centre of the sphere straight lines are drawn to touch the sphere, thus determining a segment of the sphere: find the area of the curved surface of this segment.

F. *Roorkee Engineer: Entrance.*

21. At what distance from the surface of a sphere must an eye be placed to see one-sixteenth of the surface?

22. Find the convex surface of a slice 2 ft. high cut from a globe of 17 ft. radius.

G. *Roorkee Upper Subordinate: Entrance.*

23. A sphere is 80 ft. in diameter: find what fraction of the whole surface will be visible to an eye placed at a distance of 41 ft. from the centre.

H. *Staff College.*

24. A hill, rising out of a plane in the shape of a portion of the surface of a sphere, is 300 ft. in height and 1200 ft. in diameter of base: find, to the nearest square foot, by how much its surface exceeds the area of its base. ($\pi = 3{\cdot}14159$.)

Zones of Spheres.

A. *Bombay University: L.C.E. Second Exam.*

25. A spherical zone is 4 ft. thick, and the diameters of its opposite faces are 12 and 8 ft.: find the convex surface.

B. *Calcutta University: F.E. Exam.*

26. The radius of a sphere is 5 ft. If a section be cut off by two parallel planes, so that the radii of the ends are 4 ft. and 3 ft. respectively, find the area of the curved surface thus cut off, (1) when the sections are on the same side of the centre, (2) when on opposite sides of it.

C. *Roorkee Engineer: Final.*

27. A cylindrical tower 24 ft. in diameter and 30 ft. high is capped with a hemispherical dome. The top of the dome is cut off, and over the orifice formed is built a cylindrical lantern 8 ft. in diameter and 10 ft. high, closed at the top by a plane surface. Find, in square yards, the total exterior of the building.

Additional Examination Questions—XXXV.

28. Find the percentage of error in the following approximation: The weight of a 10-in. shell, 1 in. thick, is taken as equal to that of a plate of the same material and thickness, with an upper surface area equal to that of a sphere of 8 in. diameter. (Madras University: B.E. Exam.)

29. A sphere is 100 ft. in diameter: find what fraction of the whole surface will be visible to an eye placed at a distance of 80 ft. from the centre. (Roorkee Engineer: Final.)

30. Assuming the dome of the Thomason College to be hemispherical, having a radius of 18 ft., and the superincumbent portion to be cylindrical, the exterior diameter of which is 6 ft., calculate the cost of gilding the exposed portion of the dome at 1 an. per square inch. (Do not calculate for the superincumbent portion.) (Roorkee Engineer: Final.)

31. The height of a zone of a sphere is $2\frac{1}{2}$ ft., and the diameter of the sphere is $6\frac{1}{2}$ ft.: find the area of the curved surface. (Roorkee Upper Subordinate: Monthly.)

32. Show that the area of the curved surface of a segment of a sphere exceeds that of the base by the area of a circle whose radius equals the height of the segment. (Roorkee Engineer: Final.)

CHAPTER XXXVI.

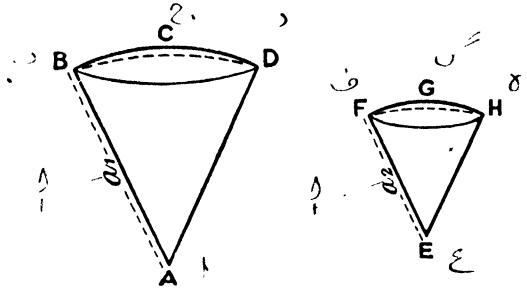
ON SIMILAR SOLIDS.

PROPOSITION XLVIII.

201. *Having given the lengths of two corresponding lines drawn in two similar solids, and the area of the surface of one of these solids, to find the area of the surface of the other solid.*

Let $ABCD$, $EFGH$ be two similar solids.

Let the corresponding lines AB and EF measure a_1 and a_2 of the same linear unit respectively.



Let the area of the surface of the solid $EFGH$ measure S_2 of any square unit.

It is required to find the area of the surface of the solid $ABCD$ in terms of a_1, a_2, and S_2.

It can be proved that the surfaces of similar solids have to one another the ratio of the squares of the lengths of any two corresponding lines that may be drawn in them.

$\therefore$ surface of solid $ABCD$: surface of solid $EFGH = AB^2 : EF^2$

That is—

$$\text{Surface of solid } ABCD : S_2 = a_1^2 : a_2^2.$$

Hence rule—

The area of the surface of a solid is found by taking its ratio to

the known area of the surface of a similar solid, and equating it to the ratio of the squares of known corresponding lengths in the two solids.

Or briefly—

$$\left.\begin{matrix}\text{Surface of first}\\ \text{solid : surface}\\ \text{of second solid}\end{matrix}\right\} = \left\{\begin{matrix}\text{ratio of the squares of corresponding lengths in first solid and}\\ \text{second solid}\end{matrix}\right.$$

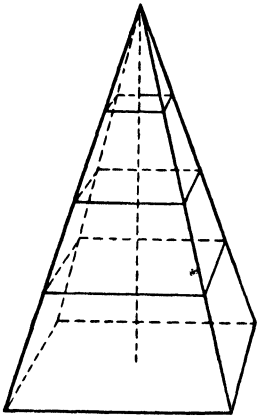
$$S_1 : S_2 = a_1^2 : a_2^2 \quad \ldots\ldots\ldots\ldots \text{(i.)}$$

Hence—

$$a_1 : a_2 = \sqrt{S_1} : \sqrt{S_2} \quad \ldots\ldots \text{(ii.)}$$

ILLUSTRATIVE EXAMPLES.

202. *Example* 1.—A pyramid is cut into four parts by three planes parallel to the base, dividing the height into four equal parts : compare the side surfaces of the four parts.



Let the side surfaces of the four parts measure A_1, A_2, A_3, A_4 of the same square unit respectively.

Then, by similar figures—

$$\left.\begin{matrix}A_1 : A_1 + A_2 : A_1\\ + A_2 + A_3 : A_1\\ + A_2 + A_3 + A_4\end{matrix}\right\} = 1^2 : 2^2 : 3^2 : 4^2 \quad \S 201.$$

$$= 1 : 4 : 9 : 16$$

$$\therefore A_1 : A_2 : A_3 : A_4 = 1 : 4 - 1 : 9 - 4 : 16 - 9$$

$$= 1 : 3 : 5 : 7$$

Example 2.—The weights of two similar solids of the same material are as 1331 : 1 : find the ratio of the surface of the first to the surface of the second.

Let V_1 cub. in. and V_2 cub. in. be the volumes of the two solids respectively.

Let A_1 sq. in. and A_2 sq. in. be the surfaces of the two solids respectively.

Let a_1 in. and a_2 in. be corresponding lengths in the two solids respectively.

Then, since the weights of bodies of the same material are proportional to their volumes—

$$V_1 : V_2 = 1331 : 1$$

$$\therefore a_1 : a_2 = \sqrt[3]{1331} : \sqrt[3]{1} \quad \ldots\ldots \S 178.$$

$$= 11 : 1$$

$$\therefore A_1 : A_2 = 121 : 1 \quad \ldots\ldots\ldots \S 201.$$

Required ratio is 121 : 1

Example 3.—The radii of the ends of a frustum of a right cone are 7 ft. and 10 ft. respectively, and the slant height is 4 ft. : if the frustum

be divided into two of equal curved surface, find the slant height of each part.

Complete the cone of which the frustum is a part.

Let DE in the figure indicate the position of the cutting plane.

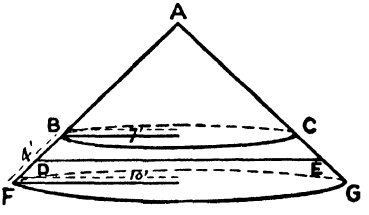
Surface of cone ABC : surface of cone ADE : surface of cone AFG $\Big\} = AB^2 : AD^2 : AF^2$. § 201.

But surface of cone $ADE = \frac{1}{2}$(surface of cone ABC + surface of cone AFG)

$\therefore AD^2 = \frac{1}{2}(AB^2 + AF^2)$

Or, if $AB = s_1$ ft.
and $AD = s_2$ ft.

$$s_2^2 = \tfrac{1}{2}(s_1^2 + s_1 + 4)$$
$$= \tfrac{1}{2}(2s_1^2 + 8s_1 + 16)$$



Again, by similar figures—

$s_1 : s_1 + 4 = 7 : 10$ . . . § 66.

$$s_1 = \tfrac{28}{3} = 9\cdot333$$

Hence—

$$s_2^2 = \tfrac{1}{2}\{2(\tfrac{28}{3})^2 + 8(\tfrac{28}{3}) + 16\}$$
$$= 11\tfrac{92}{9}$$
$$s_2 = 11\cdot508$$

and $BD = (11\cdot508 - 9\cdot333)$ ft. $= 2\cdot17$ ft.

$DF = (4 - 2\cdot17)$ ft. $= 1\cdot82$ ft.

Examples—XXXVI.

1. The radii of two spheres are as 5 : 2 : find the ratio of their surfaces.

2. The heights of two similar cones are 8 in. and 7 in. respectively: find the ratio of their curved surfaces.

3. The diagonals of two cubes are as 5 : 8 : express the surface of the first as a decimal of the surface of the second.

4. The volumes of two similar solids are as 27 : 64 : find the ratio of their surfaces.

5. The areas of two similar solids are as 49 : 81 : find the ratio of their volumes.

6. The weights of two similar solids of the same material are as 125 : 1 : express the surface of the second as a vulgar fraction of the surface of the first.

7. A cone is cut into two parts by a plane parallel to the base: if the cutting plane passes through the middle point of the height of the cone, find the ratio of the curved surfaces of the two parts.

8. A cone whose height is 2 ft. is cut by a plane parallel to the base, which divides its curved surface into two equal parts : find the distance of the plane from the vertex of the cone.

9. The base of a cone is 121 sq. in. : find the area of the base of a similar cone whose volume is to that of the former as 343 : 1331.

10. A cone is cut into three parts by two planes parallel to the base and trisecting the height : compare the curved surfaces of the three parts.

Examination Questions—XXXVI.

A. *Punjab University: First Exam. in Civil Engineering.*

1. The radii of the ends of a frustum of a cone are 5 ft. and 8 ft., and the slant height is 4 ft. : if the frustum be divided into two of equal curved surfaces, find the slant height of each part.

B. *Roorkee Engineer: Entrance.*

2. The diagonal of a cube is 2 ft. 4 in. : find the exterior surface of a second cube whose diagonal is equal to the edge of the first cube.

3. The surface of a certain solid is three times as great as the surface of a similar solid : find the proportion which the volume of the first bears to the volume of the second.

CHAPTER XXXVII.

MISCELLANEOUS EXAMPLES.

(Take $\pi = \frac{22}{7}$, unless otherwise stated.)

1. GIVE general formulæ for—area of a regular hexagon; area of a segment of a circle; area of the curved surface of a frustum of a cone; area of the curved surface of a segment of a sphere; volume of a wedge; volume of a zone of a sphere.

2. Give general formulæ for the area of a triangle, and for the surfaces and the volumes of a cone and frustum of a pyramid.

3. Find the areas of a square and a circle, the perimeter of each being 3000 ft.

4. In a right-angled triangle the sides forming the right angle are 24 ft. and 45 feet: find the perpendicular from the right angle on the hypotenuse.

5. In a parallelogram the perpendiculars between the two pairs of parallel sides are 65 ft. and 91 ft.: if one side is 119 ft., find the adjacent side.

6. The chord of an arc is 48 in., and the chord of half the arc is 27 in.: find approximately the length of the arc.

7. Give general formulæ for finding the areas of a circle, a trapezoid, a sector of a circle; also for the solidity of a sphere and a prismoid.

8. What is the area of a triangle, the sides of which are 6, 5, and 3 in.?

9. Find the length of the side of an equilateral triangle inscribed in a circle whose radius is 1 yd.

10. Give the rules for finding the areas of a parallelogram, a trapezoid, a segment of a circle, and an ellipse, and the solid content of a pyramid and a sphere.

11. The top of a circular table is 7 ft. in diameter and 1 in. thick: find its cubic contents, and the cost of polishing its upper surface at 8 annas per square foot.

12. The perimeter of a semicircle is 100 ft.: find the radius.

13. How many cubic feet of deal are contained in 200 planks, each 15 ft. long, 10 in. wide, and $1\frac{1}{2}$ in. thick?

14. The two parallel sides of a trapezoid measure 58 yds. and 42 yds., and the other sides are equal, each being 17 yds.: find the area.

15. State the rules for finding the surface and volume of a cylinder and the surface of a sphere. Give the prismoidal formula.

16. The area of a sector of a circle is 15 ft., and the length of the arc is 5 ft.: find the radius of the circle, and the number of degrees in the arc. ($\pi = 3{\cdot}14159$.)

17. What is the weight of $2666\frac{2}{3}$ cub. ft. of a stone, a piece of which 20 in. long, 8 in. broad, and 15 in. deep weighs 280 lbs.?

18. Give rules for finding the areas of a triangle, a circle, and a sector of a circle.

19. The sides of a triangle being 16, 17, and 18 ft., determine whether its area is greater or less than that of an equilateral triangle whose sides are each 17 ft.

20. Find the area of the triangle whose sides are $4\frac{1}{2}$, 6, and 8 ft. to the nearest square inch.

21. The wheel of a carriage, in passing over 25,600 in., makes as many revolutions as there are inches in its circumference: find its diameter.

22. Find to the nearest foot the dimensions of a triangle whose sides are in the ratio of 5 : 6 : 8, and whose area is one acre.

23. Two adjacent sides of a quadrilateral are 3 ft. and 4 ft. respectively, and the angle contained by them is 90°; the other two sides of the quadrilateral are equal, and the angle contained by them is 60°: find the area.

24. Find the weight of cast iron in a pillar, the interior and exterior diameters being 9 in. and 11 in. respectively, and length 10 ft., a cubic inch of cast iron weighing 0·26 lb.

25. Two tangents drawn from an external point to a circle are at right angles, and measure $1\frac{3}{4}$ in. each: find the area of the circle.

26. The extremity of the minute-hand of a clock moves 5 in. in 3 minutes: what is its length?

27. The circumference of a circular field is 314 yds. 10·285714 in.: what is its diameter and area?

28. How many 3-in. cubes can be cut out of a cubic foot?

29. The sides of a triangle are 55, 48, and 71 ft. respectively: calculate its area in square feet to seven places of decimals.

30. The perimeter of an equilateral triangle is 200 ft.: find its area.

31. If you had to make a pathway round a quadrangle, so that its area may be just half that of the quadrangle, the dimensions of the latter being 350 ft. × 150 ft.: what would be the width of the pathway?

32. A pontoon with hemispherical ends is 50 ft. in extreme length, and diameter 5 ft.: find its capacity.

33. Find the length of the arc and area of the segment of a circle, having given the chord of the arc = 40, and chord of half the arc = 25.

34. An oblong room is 21 ft. 7 in. long, 15 feet wide, and 10 ft. high. In it are two doors 7 ft. × 3 ft., two windows 5 ft × 3 ft., with semicircular heads, and a fireplace 4 ft. × 3 ft. 6 in. What is the surface of the walls?

35. Find in acres the area of a triangle whose sides are 32 chains 11 links, 25 chains 32 links, and 22 chains 75 links respectively.

36. The area of a trapezoid is $3\frac{1}{8}$ acres, the sum of the two parallel sides is 242 yards: find the perpendicular distance between them.

37. A ladder is 25 ft. long, and stands upright against a wall: find how far the bottom of the ladder must be pulled out from the wall so as to lower the top 1 ft.

38. The diagonals of a rhombus are 44 and 117 ft. respectively: find the area, length of side, and height of the rhombus.

39. Compare the areas of two regular pentagons, one inscribed in a given circle, and the other described about it.

40. Compare the areas of the circles inscribed within and circumscribed about an equilateral triangle.

41. The base of an aquarium is a square, the height is half a side of the base, and there is no lid; the glass cost Rs.31 4 annas at Rs.15 per square yard: find the number of gallons the aquarium will hold.

42. If a cubic inch of gunpowder weigh $\frac{1}{2}$ oz., what weight of powder would be required to fill a conical vessel 8 in. in diameter and slant height 5 in.?

43. The area of a regular hexagon is 400: find the length of its side, and the radii of its inscribed and circumscribed circles.

44. How many yards of canvas 27 in. wide are required to construct a conical tent 18 ft. in diameter and 13 ft. in height?

45. The sides of a triangle, of which the perimeter is 594 ft., are in the ratio 13 : 20 : 21 : find its area.

46. Find the area of a quadrilateral field *ABCD*, the side *AB* being 457 ft., *BC* 568 ft., *CD* 570 ft., *DA* 807 ft., and *AC* 793 ft.

47. A tower on the bank of a river is 120 ft. high, and the angle of elevation of the top from the opposite bank is 30°: find the breadth of the river.

48. A parallelogram whose adjacent sides are severally M and N units of length can be divided into MN parallelograms, each having adjacent sides of unit length: can the area of the parallelogram be taken as MN units of area?

49. The radius of a circle being 6 in., find the area of the inscribed regular hexagon.

50. Find the length in yards of a cord fastened to a stake at one end and to a cow's neck at the other, so as to allow her to feed on a bigha of grass and no more. ($\pi = 3{\cdot}14159$.)

51. A complete pile of shot stands upon a rectangular base whose unequal sides contain 6 and 14 shot respectively: find the number of shot in the pile.

52. What must be the side of an equilateral triangle so that its area may be equal to that of a square of which the diagonal is 180 ft.?

53. What ratio does the area of a circle bear to that of its inscribed square?

54. Find the side of an equilateral triangle of which it cost as much to pave the area at 1*s.* per square foot as to fence the sides at 6*s.* 6*d.* per foot.

55. Find the area of a triangle whose sides are 243, 324, and 405 yds. respectively, and express the result in acres, roods, and perches.

56. The diagonals of a field, the sides of which are all equal, are 88 yds. and 110 yds. respectively: find its acreage.

57. Given the perimeter equal to p linear units, find the greatest area that it will enclose.

58. In a circular riding school of 100 ft. in diameter a circular ride within the outer edge is to be made of a uniform width of 10 ft.: find the cost of doing this at 4*d.* per square foot. ($\pi = 3{\cdot}14159$.)

59. Find the area of the maximum triangle that can be inscribed in a circle of 10 ft. radius.

60. Define, "wedge," "parallelopiped," "prism," "cone," "pyramid," "prismoid," and illustrate them with sketches.

61. Lay down a field from the annexed notes, and find its area—

	Links.	
	To *D*	
	1160	
To *C* 596	1016	
To *B* 304	392	
	From ⊙ *A*	go east

62. What is the area of the slant surface of a frustum of a right cone, the areas of the two circular ends being 1256·64 sq. in. and 78·54 sq. in. respectively, and the vertical height of the frustum 20 in.? ($\pi = 3{\cdot}1416$.)

63. If in the last question 20 in. had been the height of the cone before it was truncated, instead of that of the frustum, what would be the volume of the frustum? ($\pi = 3{\cdot}1416$.)

64. What will be the expense of painting a conical church-spire at 8 annas per square yard, the circumference of the base being 64 ft., and its height 118 ft.?

65. In a right-angled triangle prove that area $= s(s - c)$, c being the hypotenuse, and s the semi-perimeter.

66. A man walking along a straight road observes at one milestone a house making an angle of 30° with the road, and that at the next milestone the angle is 60° : how far is the house from the road ?

67. Each side of a rhombus is 24 ft., and one of the diagonals is also 24 ft. : find the area.

68. The perimeter of a semicircle is 100 ft. : find the area. ($\pi = 3{\cdot}14159$.)

69. The sides of a triangle are 32, 27, and 48 in. : find the area of the triangle and the diameter of the circumscribing circle.

70. Define "cube," "cylinder," "sphere," "spheroid," and illustrate them with sketches.

71. The diameter of a circle, AF, is 15 ft. ; DF, the height of a segment whose chord, BE, is perpendicular to the diameter, is 3 ft. : join AB, and find its perpendicular distance from the centre of the circle.

72. The side of a regular octagon is 20 ft. : find the area ; also the area of the inscribed circle.

73. An equilateral triangle and a square have the same area : compare their perimeters.

74. The adjacent sides of a parallelogram are 8 ft. and 16 ft., and its area is half that of a square having the same perimeter : find the perpendicular distance between each pair of opposite sides.

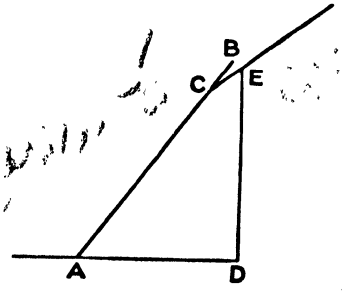
75. Find the area of a square whose side is equal to two-thirds of the radius of a circle whose area is 9900 sq. ft.

76. The area of a sector is 99 sq. ft., and the length of the arc 9 ft. : find the radius and the angle at the centre. ($\pi = 3{\cdot}1416$.)

77. Find the area of a segment of a circle whose chord is 24 in. and height 5 in.

78. Draw a plan of the field *ABCDE*, and calculate its area from the accompanying notes—

Links.	
To ⊙ *D*.	
900	
To *E* 300 — 600	
500	o *C*
200	400 to *B*
From ⊙ *A*	



79. Give general formulæ for finding the volume of—

(*a*) Pyramid ;
(*b*) Oblate spheroid ;
(*c*) Prolate spheroid.

80. The chord of an arc is 15·78 ft., and the height of the arc is 2·8 ft. : find the diameter of the circle.

81. *AB* (see sketch) represents an iron ladder resting against the eaves of a thatched building. The eaves project 2 ft. from the walls, and the slope of the roof is 45°. If $CB = 2$ ft., $ED = 18$ ft., $AD = 14$ ft., how far must the foot of the ladder be pulled out in order that the top may just reach the eaves? What is the length of the ladder ?

82. The greatest and least diameters of two spheroids are 4 ft. and 3 ft. respectively, one spheroid being oblate and the other prolate: what is the volume of each spheroid?

83. Two sides of a triangle are 143 ft. and 165 ft.; the perpendicular on the third side is 132 ft.: find the area of the triangle.

84. A circle of circumference 70·65 lies entirely within another of circumference 117·75: find the area of the annulus, taking $\pi = 3{\cdot}14$.

85. Find the area of the space bounded by two concentric circles and their radii in terms of the intercepted arcs and the distance between them.

Two concentric circles have radii 10 ft. and 15 ft. respectively: calculate the area of the figure bounded by these circles and radii inclined to each other at an angle of 40°. ($\pi = 3{\cdot}14159$.)

86. In a right-angled isosceles triangle the radius of the inscribed circle is 1 ft.: find the sides.

87. The area of a sector is 230 sq. ft.; the angle of the sector is 40°: find the whole perimeter of the sector. ($\pi = 3{\cdot}14159$.)

88. A polygonal field is measured by carrying a base-line from one corner, *A*, to another, *B*, and taking offsets right and left, as in the following field-book:—

	Yards.	
	To *B*	
	250	*B*
40	220	
	180	60
90	160	
	150	70
	90	20
30	70	
	50	15
	From ⊙ *A*	go north-west

Find the area.

89. Give general formulæ for finding the volumes of—

(*a*) Frustum of cone;

(*b*) Segment of sphere.

90. Find the cost of turfing a grass plot in the form of a regular octagon at the rate of 2 annas per square yard, each side of the octagon being 20 ft.

91. The surface of a sphere is 113·0976 sq. in.: find its diameter and solid content. ($\pi = 3{\cdot}1416$.)

92. The angle contained by two sides of a triangle is 30°, and the length of these sides are 215 ft. and 248 ft. respectively: find the area.

93. The diagonals of a quadrilateral are 30 and 40 chains, and they intersect at an angle of 45°: find the area.

94. Each side of a rhombus is 32 ft., and each of the larger angles is equal to twice each of the smaller angles: find the area.

95. Find the diameter of the circle described round the triangle having the following sides: 136, 125, 99.

96. What is the area of a sector greater than a semicircle, the chord of the whole arc of the remaining sector being 72 ft., the chord of half the arc 45 ft., and the radius 37 ft. 6 in.?

97. Draw a rough sketch of the field and calculate its area from the following notes:—

	Links.	
	To ⊙ *F*	
	1280	
	960	280 to *C*
To *E* 470	400	
	Turn to	the right
	To ⊙ *B*	
	1120	
To *D* 60	600	
	320	600 to *C*
	From ⊙ *A*	go west

98. A pendulum swings through an angle of 30°, and the end describes an arc of $13\frac{2}{31}$ in. : find the length of the pendulum.

99. The equidistant ordinates of a curvilinear area are 1, 41, 58, 61, 61, 57, 54, 46, 41, 33, 24, 16, 1, and they are measured from a base-line 240 ft. in length : find approximately the area.

100. Required the volume of a rectangular parallelopiped which is 8 ft. 9 in. long, 5 ft. 6 in. broad, and 4 ft. 3 in. high. Find also the length of its diagonal.

101. The area of a circular table is 805 sq. in. : find how many nails, each $\frac{1}{2}$ in. apart, will be required to nail on a border. ($\pi = 3{\cdot}1416$.)

102. The bottom of an axle round which a well rope is wound is 2 ft. 6 in. from the surface of the ground, and 9 in. in diameter. When all the rope is wound up, it takes fourteen turns of the handle before the rope can be lowered so as to just touch the bottom of the well. What is the depth of the well from the surface of the ground ?

103. A cylindrical pillar has a hemispherical top. Diameter of base 4 ft., total height of pillar 10 ft. Find its cubic content.

104. A log of English oak is 15 ft. long, 18 in. broad, and 12 in. thick : at what distance from one end must the log be cut that the smaller portion may weigh 5 cwt., supposing 1 ton of English oak to contain 36·205 cub. ft. ?

105. The sides of a quadrilateral taken in order are 27, 36, 30, and 25 ft. respectively, and the angle contained by the first two sides is a right angle : find the area.

106. A man standing due south of a tower observes its altitude to be 60° ; after walking 150 ft. in an easterly direction, he finds its altitude to be 45° : find the height of the tower.

107. Find by duodecimals the volume of a rectangular parallelopiped having the following dimensions : 7 ft. 5 in., 6 ft. 7 in., 3 ft. 10 in.

108. Draw a plan and calculate the area of a field from the following measurements in links :—

	To *A*	
	1700	
From	*C*	range to *A*
	To *C*	
	800	
	400	65
From	*B*	go north
	To *B*	
	1500	
	1100	180
	625	240
From	*A*	go east

109. A gardener wishes to make a grass plot in the form of a regular hexagon that shall contain 260 sq. yds. : what must be the length of its side ?

110. Find the area of a field, one side of it being 198 links and 7 ordinates to it, measured at equal distances to the opposite curvilinear boundary, being in order 60, 75, 80, 82, 76, 63, and 50 links.

111. What is the weight of an iron shell, the external and internal diameters of which are 9 in. and 6 in. respectively, if an iron ball of 4 in. diameter weigh 9 lbs. ?

112. A pillar 60 ft. high has an elliptical base and top. The major axes of base and top are 20 ft. and 10 ft. respectively, the minor axes 8 ft. and 4 ft. respectively. Find its cubic content.

113. Find the area of a pentagonal field $ABCDE$, given that $AB = 20$ yds., $BC = 41$ yds., $CD = 51$ yds., $DE = 100$ yds., $AE = BD = 58$ yds., and that AB and DE are parallel.

114. Find, correct to two places of decimals, the area of a regular octagon inscribed in a circle of unit radius.

115. The difference between the diameter and the circumference of a circle is 10 ft. : find the diameter ($\pi = 3{\cdot}1416$.)

116. The minute-hand of a clock makes an arc of 11 in. in ten minutes : find the radius of the face of the clock. ($\pi = 3{\cdot}1416$.)

117. The height of an arc is 1 ft. 3 in., and the diameter of the circle is 11 ft. 3 in. : find the chord of half the arc.

118. Find the area of a field, one side of it being 990 links, and seven equidistant ordinates from it to the opposite curvilinear boundary being 300, 375, 400, 410, 380, 315, 250 links.

119. The diameter of the base of a cone is 3 ft. 4 in., and its slant side is 16 ft. : what is its solidity ? ($\pi = 3{\cdot}1416$.)

120. An iron pipe is 3 in. in bore, $\frac{1}{2}$ in. thick, and 20 ft. long : find its weight, supposing that a cubic inch of iron weighs 4·526 ozs.

121. Supposing iron to be eight times the weight of oak, what will be the diameter of an iron ball whose weight is equal to that of a ball of oak 18 in. in diameter ?

122. Two sides of a triangle are 40 and 60 yds., and they contain an angle of 30° : find the area.

123. Three sides of a quadrilateral field taken in order are 15, 10, and 20 chains ; the angle between the first two sides is 150°, and the angle between the second and third is 60° : find the area in square chains.

124. A square field is bounded by a path 3 yds. wide, the field and path together occupying $2\frac{1}{2}$ acres : find the cost of covering the path with gravel at 1*s.* 6*d.* per square yard.

125. Two circles whose radii are 30 ft. and 40 ft. intersect ; the distance between their centres is 50 ft. : find the length of their common chord.

126. The sides of a triangle are 111, 175, and 176 ; two straight lines are drawn across the triangle parallel to the largest side, and dividing each of the other sides into three equal parts : find the areas of the three parts into which the triangle is divided.

127. Apply Simpson's rule to find approximately the area of a curvilinear figure from the following data : ordinates 0, 9, 13, 17, 19, 22, 17, 13, 8, 4, 2 ; base = 61.

128. The height of a cylinder is to be equal to the radius of the base, and the volume is to be 500 cub. in. : find the height.

129. Supposing a cubic foot of brass to weigh 8500 ozs., find the weight of a yard of brass wire, the thickness of which is $\frac{1}{36}$ in.

130. The base of a pyramid is a rectangle which is 18 ft. by 26 ft. ; the length of the straight line drawn from the vertex to the middle point of either of the shorter sides of the base is 24 ft. : find the volume.

131. What number of 8-in. cannon balls can be made out of 100 tons of iron, supposing a cubic foot to weigh 441 lbs. ? ($\pi = 3{\cdot}1416$.)

132. The sides of a quadrilateral field $ABCD$ are $AB = 20$ yds., $BC = 26$ yds. 2 ft., $CD = 80$ yds., $DA = 85$ yds. 2 ft. ; the shorter diagonal $AC = 33$ yds. 1 ft. : prove that the angles ABC, ACD are each of them a right angle, and calculate the area of the field.

133. The perimeter of one square exceeds that of another by 100 ft., and the area of the larger square exceeds three times the area of the smaller by 325 sq. ft. : find the length of their sides.

134. A gravel walk of uniform breadth is made round a rectangular grass plot, the sides of which are 20 and 30 yds. : find the breadth of the walk if its area be three-tenths of that of the grass plot.

135. The chord of an arc is 6 in., and the radius of the circle is 9 in. : find the arc.

136. An estate which has been surveyed is one hundred million times as large as the plan which has been made of it : express the scale of the plan in terms of inches to a mile.

137. A right-angled triangle, of which the sides are 5 and 12 in. in length, is made to turn round its hypotenuse : find the surface of the double cone thus produced.

138. Find the volume of a cylindrical shell, the radius of the inner surface being 12 in., and the thickness 3 in., and the length 10 ft. ($\pi = 3{\cdot}1416$.)

139. What is the volume of a spherical zone, the diameters of its ends being 10 and 12 in., and its height 2 in. ? ($\pi = 3{\cdot}1416$.)

140. The height of a right circular cylinder is 4 ft. : find the height of a similar cylinder of nine times the volume.

141. The parallel sides of a trapezoid are 100 and 180 ft., and the angles which the other two sides make with the shorter side are 135° and 150° : find the area in square yards.

142. $ABCDEF$ is a figure having six equal sides ; $AB = 57{\cdot}8$ ft., $BF = 64{\cdot}4$ ft., and the portion $BCEF$ forms a rectangle : find the area.

143. The side of a square is 85 yds., and a path 10 yds. wide goes round the square outside it : find how many stones, each 1 ft. 4 in. long by 10 in. wide, will be required to pave the path.

144. If the length of the tangents drawn from an external point to a circle be 21 in., and the angle between them be 60°, show that the area of the circle is 462 sq. in. nearly.

145. The base of a field in the form of a trapezoid is 30 and the two perpendicular sides are 28 and 16 chains respectively : how would you divide it equally between two persons by a fence parallel to the perpendiculars ?

146. A solid is composed of a hemisphere and a cone on opposite sides of the same circular base ; the diameter of this base is 5 ft., and the height of the cone is 5 ft. : find the volume of the solid. ($\pi = 3{\cdot}1416$.)

147. Find by duodecimals the volume of a rectangular solid which is 8 ft. 9 in. long, 5 ft. 6 in. broad, and 4 ft. 3 in. deep.

148. A square tower, 21 ft. on each side, is to have either a flat roof covered with sheet lead which costs 6*d.* per square foot, or a pyramidal roof whose vertical height is 10 ft., covered with slates which cost 18*s.* 9*d.* per hundred, and each of which has an exposed surface of 12 in. × 9 in. : find the cost in each case.

149. Find the contents of a wedge whose base is 16 in. long and $2\frac{1}{4}$ in. broad, its height being 7 in. and its edge $10\frac{1}{2}$ in.

150. The base of a complete triangular pile of shot is an equilateral triangle whose side contains ten shot : find the number of shot in the pile.

151. The volume of a cone is equal to the area of the base by one-third the perpendicular height. Deduce the volume of a frustum of a cone, the diameters of the ends being d and d', and the altitude h.

152. Find the number of cubic feet of masonry in an arch of the following dimensions: span = 60 ft. = radius of inner curve, depth of arch = 4 ft., and length = 20 ft.

153. A rope-dancer, walking on a slack rope 153 ft. long, fastened to the top of two perpendicular poles, each 80 ft. high, placed at a distance of 147 ft. from each other, breaks the rope and falls at a distance of 48 ft. from one of the poles: from what height did he fall?

154. The side of an equilateral triangle is 1 ft.; each of the sides is divided into four equal parts, and the nearest points of division are joined: find the area of the hexagon so formed.

155. Determine the scale used in the construction of a plan upon which a square foot of surface represents an area of 10 acres.

156. What is the solidity of a cube whose diagonal is 81 ft.?

157. The altitude of a cone is 10 ft., and the diameter of its base 1 ft. Divide it into three equal parts by sections parallel to the base, and determine the altitude of its three parts.

158. Find the number of cubic feet that must be removed to form a prismoidal cavity, depth 12 ft.; top and bottom are rectangles, the corresponding dimensions of which are 400 ft. × 180 ft. and 150 ft. × 350 ft.

159. Find the weight of a solid ring of wrought iron whose external diameter is 1 ft., thickness of iron 2 in. (Wrought iron weighs 0·28 lb. per cubic inch.)

160. Explain the meaning of the prismoidal formula by taking an example.

161. A cast-iron tank is required to hold 8000 galls. of water: what is the area of each side and the area of the base of an octagonal one, the height of which must be 8 ft.?

162. Find the area of a rectangle inscribed in a triangle of sides 13, 37, 40, having one side coincident with the longest side of the triangle, and the ratio of adjacent sides 10 : 1.

163. A hall can be paved with two hundred square tiles of a certain size; if each tile were 1 in. longer each way it would take 128 tiles: find the length of each tile.

164. A portion of a circle is cut off by two parallel chords situated on the same side of the centre, and of lengths $\sqrt{3}+1$ and $\sqrt{3}-1$ respectively, the perpendicular distance between them being 1 ft.: find the radius of the circle.

165. The fence of an enclosure in the form of a regular octagon cost £840 at 4*s.* 6*d.* per foot: what will be the cost of gravelling the surface at 10½*d.* per square yard.

166. A sphere of copper 12 in. in diameter is beaten out into a circular plate 40 in. in diameter: what is the thickness of the plate, supposing 5 per cent. of the metal is lost in working?

167. Find to the nearest square inch the quantity of leather required to cover a spherical football which measures 23 in. in circumference.

168. Find the area in square inches of the exposed surface of three bricks each 9 in. × 4½ in. × 3 in., piled as follows: The first is laid flat on a table, the second is placed on end on the first and at right angles to it, the third is laid flat on the second at right angles to it.

169. The interior diameter of a cylindrical ring is 26 in., and its thickness 8 in.: what is its solidity?

170. The ends of a prismoid are rectangles the corresponding dimensions of which are 8 ft. × 7 ft. and 10 ft. × 6 ft., and the height is 4 ft.: find the volume.

171. The diameter of a 12-lb. shot is 4½ in.: what is the thickness of an 8-in. shell which weighs $43\frac{69}{81}$ lbs.?

172. The frame of a looking-glass is 3 ft. 9 in. long and 2 ft. 4 in. broad: find the dimensions of the glass when the area is equal to that of the frame.

173. Three equal circles touch each other : find a formula for the area of the space between them, r being the radius of the circles.

174. At what time between 10 and 11 o'clock are the hands of a watch (i.) coincident, (ii.) opposite one another, (iii.) at right angles, (iv.) twenty-five divisions apart ?

175. The side of a square is 12 ft. : the square is divided into three equal parts by two straight lines parallel to the diagonal : find the perpendicular distance between the parallel straight lines.

176. Two spheres, whose radii are $7\frac{1}{2}$ in. and $5\frac{1}{4}$ in. respectively, are melted down and cast into a hollow shell : if the external diameter of this shell be 3 ft., find the diameter of the internal hollow space.

177. A solid cylindrical projectile having one end hemispherical, the other flat, is 4 ft. long and 6 in. in diameter : find its volume and whole surface.

178. The edge of a wedge is 4 ft. 6 in., the length of the base 2 ft. 8 in., breadth of the base 1 ft. 4 in. ; the height of the wedge is 21 in. : find the volume.

179. How many marbles, each 1 in. in diameter, can be packed in a box whose internal dimensions are an exact cubic foot ?

180. How much canvas will make a conical tent 11 ft. in height and 12 ft. in diameter at the base, width of canvas being 45 in. ?

181. Two maps are of the same size. On the first a line 8·56 in. in length represents 128·4 miles ; on the second an area of 100 acres is represented by $\frac{1}{480}$ sq. in. Compare the areas represented by the two maps.

182. If a cubic foot of iron weigh 4 cwt., what will be the weight of a water-pipe of that material, the length of which is 10 ft. 4 in., the interior diameter 8 in., and the thickness of metal $\frac{1}{2}$ in. ? Also what will be the cost of 2 miles of such pipe at £5 per ton ?

183. If a cubic foot of metal weigh 4 cwt. 1 qr., and is worth Rs.280 per ton, what will be the cost of 1 mile of piping made out of it, with a 9-in. bore and $\frac{3}{4}$ in. thick ? ($\pi = 3{\cdot}1416$.)

184. A maypole was broken by the wind, and its top struck the ground 20 ft. from the base. Had it been broken 5 ft. lower down, its top would have extended 10 ft. further from the base : required the height.

185. Three men bought a grindstone of 50 in. diameter, each paying one-third part of the expense : what part of the diameter must each person grind down for his share ?

186. How many cubic feet are there in a grinding stone which is 3 ft. in diameter, 6 in. thick at the circumference, and 9 in. at the centre ?

187. A cone 40 in. high and 17 in. diameter is to be cut into three equal portions by planes parallel to the base : what must be the altitude of each part ?

188. An embankment 100 yds. long is uniformly 40 yds. wide at the bottom ; it is 12 ft. deep at one end, and gradually increases to 15 ft. deep at the other, and the upper widths at these ends are respectively 76 ft. and 85 ft. : find the cubic yards of embankment.

189. A vessel is in the shape of a cube ; it is without a lid : if the external length is 3 ft., and the thickness of the material 1 in., find the number of cubic inches of material.

190. The height of a solid 6-in. cube is diminished by pressure to $5\frac{5}{8}$ in. : suppose the lateral expansion uniform throughout the mass, what will be the dimension of the new base of this solid ?

191. A pyramidal roof 16 ft. high, standing on a square base which is 24 ft. on each side, is covered with sheet lead $\frac{1}{18}$ in. thick : how many bullets will the lead make, each in the form of a cylinder $\frac{1}{2}$ in. long and $\frac{6}{11}$ in. in diameter, terminated at one end by a cone of the same diameter and $\frac{3}{4}$ in. high ?

192. A conical glass 6 in. in diameter and 4 in. high is filled with water, and a spherical ball 4 in. in diameter is sunk into it as far as it will go: find the weight of water displaced, taking the weight of a cubic foot of water at $62\frac{1}{2}$ lbs.

193. A cutting and an embankment have to be made, the former 30 ft. deep, the latter half as high. The top of the embankment and bottom of the cutting are equal and 40 ft. broad, and the sides of both slope at the same angle, 45°. Not allowing for the expansion of the excavated earth, what length of embankment will 110 yards' run of cutting make?

194. A bridge arch has a span of 60 ft., a rise of 10 ft., and a depth of 4 ft., and its length from face to face is 30 ft.: find how many cubic feet of masonry are in the arch.

195. Deduce the volume-formulæ of a prism, a cylinder, a pyramid, and a cone from the volume-formula of a prismoid.

196. A flagstaff stands on a tower. I measure from the bottom of the tower a distance of 100 ft. I then find that the top of the flagstaff subtends an angle of 45°, and the top of the tower an angle of 30° at my place of observation. What is the height of the flagstaff?

197. The areas of the floors of two rooms are the same, but the volume of one room is 1800 cub. ft. greater than that of the other. The length and the height of the larger room are $21\frac{3}{4}$ ft. and 15 ft. respectively, the width and the height of the smaller 15 ft. and 10 ft. Find the remaining dimensions.

198. What will be the cost, to the nearest rupee, of arching over a room 32 ft. long and 20 ft. span, the arch being segmental, with a rise of $\frac{1}{4}$ of span, and thickness 9 in.? (Cost of masonry, Rs. 35 per 100 cub. ft.)

199. A cylinder 5 ft. long and 3 ft. in diameter is closed by a hemisphere at each end: find the area of the whole surface. ($\pi = 3{\cdot}1416$.)

200. Find the number of gallons of water required to fill a tank, the depth of which is $4\frac{1}{2}$ ft., and the top and bottom of which are rectangles, the corresponding dimensions of which are 250 ft. × 16 ft. and 240 ft. × 14 ft.

201. The cross-section of a brick subway 20 ft. long is a rectangle surmounted by a semicircle. The total height is 8 ft. and breadth 4 ft., both exclusive of the bricks; the thickness of the bricks is $4\frac{1}{2}$ in. Find the weight of the bricks, if a brick containing $\frac{9}{160}$th of a cubic foot weigh 5 lbs.

202. The height of a frustum of a cone is 7 ft., and the diameters of the two ends are 8 ft. and 10 ft. respectively; the frustum is cut into two pieces of equal volume by a plane parallel to the ends: find the distance of this plane from the smaller end.

203. A bridge arch has a span of 20 ft., a rise of 3 ft., depth of voussoir 2 ft., and its length from face to face is 30 ft.: find how many cubic feet of masonry it contains.

204. An observer holding a foot rule vertically before him at a distance of 3 ft. from his eye, finds that a distant flagstaff is exactly covered by $2\frac{3}{4}$ in. of the rule. He then advances 100 ft. directly towards the staff, and observes that, still holding the rule outstretched at the same distance, the staff is covered by 3 in. of the rule. Find the height of the flagstaff, its foot being constantly on a level with the observer's eye.

205. The carpeting of a room twice as long as it was broad, at 5*s.* per square yard, cost £6 2*s.* 6*d.*, and the painting of the walls at 9*d.* per square yard cost £2 12*s.* 6*d.*: find the dimensions of the room.

206. The content of a cistern is the sum of two cubes whose diagonals are 10 and 2 in., and the area of its base is the difference of two squares whose sides are $1\frac{1}{2}$ and $1\frac{3}{4}$ ft.: find the depth of the cistern.

207. A frustum of a circular cone is trimmed just enough to reduce it to a frustum of a pyramid with square ends: find what fraction of the volume is removed.

208. The area of the surface of a sphere is 25 sq. in. : find the volume. ($\pi = 3{\cdot}1416$.)

209. Find the number of cubic feet which must be removed to form a prismoidal cavity; the depth is 12 ft., and the top and the bottom are rectangles, the corresponding dimensions of which are 400 ft. by 180 ft., and 350 ft. by 150 ft.

210. Every edge of a certain triangular prism measures 10 in.: find the volume.

211. A railway tunnel is 21 ft. wide in the clear, 12 ft. high to the springing, and has a semicircular arch; the foundations are 1 ft. 6 in. deep and 2 ft. thick; the thickness of the side walls is 1 ft. 6 in., and that of the arch 1 ft. How many cubic feet of brickwork are there in 100 ft. in length of this tunnel?

212. A circular chimney tapers from base to top. It is 14 ft. in diameter at base, 8 ft. 9 in. in diameter at top, and 57 ft. high. The inside of the shaft is circular, and uniformly 7 ft. in diameter. What is the solid content of the chimney?

213. A person observes the elevation of a tower to be 60°, and on receding from it 100 yds. further, he finds the elevation to be 30°: required the height of the tower.

214. Show that the chord of a quadrant of a circle divides the circle into two parts, the areas of which are in the ratio of 10 to 1 nearly.

215. The radius of a circle is 15 ft. : find the areas of the two parts into which it is divided by a chord equal to the radius. ($\pi = 3{\cdot}1416$.)

216. A sphere 15 in. in diameter is divided into three parts of equal height by two parallel planes: find the volume of each.

217. What is the volume of a prismoid, the length and breadth of its greater end being 24 and 16 in., and those of its top 16 and 12 in., and its height 120 inches? Answer to be given in cubic feet.

218. An incomplete pile of shot stands upon a square base whose side contains 20 shot, and there are ten courses: find the number of shot in the pile.

219. The radii of the ends of a frustum of a right circular cone are 7 ft. and 8 ft. respectively, and the height is 3 ft.: find the volumes of the two pieces obtained by cutting the frustum by a plane parallel to the ends and midway between them.

220. The base of a right prism is a rectangle which measures 7 in. by 8 in.; find the volume of the solid obtained by cutting off a piece of this prism, so that the sum of the four parallel edges is 42 in.

221. Find the quantity of masonry in a roof arch, and its cost at Rs. 35 per 100 cub. ft. Dimensions: length of arch, 40 ft.; span, 15 ft.; rise, 3 ft.; and thickness, 18 in.

222. The diameters of the ends of a frustum of a cone are 12 ft. and 16 ft. respectively, and the height of the frustum is 6 ft.; the frustum is divided into three equal parts by planes parallel to the ends: find the distances of the planes from the larger end.

223. Assuming that a dome is hemispherical, and the cost of whitewashing its exterior surface at 4 annas per 100 square feet is Rs. 4, and that of the interior at the same rate to be Rs. 2 8 annas, find the cost of the masonry at Rs. 23 per 100 cub. ft.

224. What will be the cost of arching over a room 26 ft. long and 18 ft. wide, the arch being segmental, with a rise of $\frac{1}{3}$ of width, its thickness $1\frac{1}{2}$ ft., and the price of masonry being Rs. 25 per 100 cub. ft.?

225. A pipe of 4 in. diameter is sufficient to supply a town with water: what must be the diameter of a pipe which, with the same velocity, will supply it when its population is increased by one-half?

226. Find the cubic content of a piece of road embankment 400 ft. long,

the longitudinal slope being regular, the height at the ends being 6 and 4 ft. respectively, the side slopes 2 to 1, and the breadth at top everywhere 30 ft., the ends being vertical.

227. In a frustum of a right circular cone the larger diameter is 4 ft. 9 in., the smaller diameter 3 ft. 6 in., and the perpendicular height 5 ft. : find the area of the whole surface.

228. A sphere of 1 ft. radius rests on a table : find the volume of the right hollow cone which can just cover it, the section of the cone through the axis being an equilateral triangle. ($\pi = 3{\cdot}1416$.)

229. The frustum of a right cone is 6 ft. high, the radius of the smaller end is 2 ft., and the radius of the larger end is 3 ft. : find the position of the plane parallel to the ends which will divide the frustum into two equal parts. Find also the volume of each part.

230. What fraction of the earth's surface can be seen from a height of 10,000 ft. ? (Diameter of the earth = 8000 miles.)

231. How many yards of rope will be required to suspend 20 punkahs, each 48 ft. long, with three hooks at 24 ft. spaces, and hung 16 ft. below the ceiling, from two hooks 24 ft. apart? A rope runs from each of the ceiling hooks to each of the punkah hooks.

232. A round bar of silver 2 ft. long and 2 in. in diameter is to be drawn out into wire of $\frac{1}{50}$ in. in diameter : what will be the length of the wire obtained ?

233. A ball of lead 4 in. in diameter is covered with gold : find the thickness of the gold in order that the volumes of gold and lead may be equal. ($\pi = 3{\cdot}1416$.)

234. In laying the foundation of a house, an excavation is made 40 ft. long, 30 ft. broad, and 6 deep. The earth removed is spread uniformly over a field containing half an acre : find how much the surface of the field will be raised

235. A pyramid has for its base an equilateral triangle of which each side is 1 ft., and its slant edge 3 ft. : required its surface and solid content.

236. Find the velocity at which a person is travelling in latitude 45° owing to the earth's rotation, assuming the earth's radius to be 4000 miles.

237. A bowl is in the shape of a segment of a sphere ; the depth of the bowl is 9 in., and the diameter of the top of the bowl is 3 ft. : find to the nearest pint the quantity of water that the bowl will hold.

238. Find in inches the diagonal of a cube whose surface is equal to a square yard.

239. The height of a frustum of a pyramid is 4 in. ; the lower end is a rectangle which is 9 by 12 in. ; the upper end is a rectangle of which the longer side is 8 in. : find the volume of the frustum.

240. The radii of the ends of a frustum of a right circular cone are 7 ft. and 8 ft. respectively, and the height is 3 ft. ; the frustum is cut into three, each 1 ft. in height, by planes parallel to the ends : find the volume of each of the pieces.

241. A pyramid is cut out from a cube (edge = a) by a plane passing through the extremities of three edges, meeting at a corner of the cube : find the area of the surface of the figure left.

242. How much would it cost to have a pit dug at 4 annas per cubic yard, its size at the top being 34 ft. 4 inches long, 30 ft. broad, the sides sloping at an angle of 45°, and its depth 13 ft. 6 in. ?

243. Investigate an expression for determining the area of a regular polygon in terms of the number of sides and the radius of the inscribed circle.

244. The lower portion of a haystack is an inverted conic frustum, and the upper part a cone; the greatest height is 30 ft., and the greatest circumference 60 ft., the height of the frustum 20 ft., and the diameter of the base 16 ft. : find the volume in cubic feet.

245. Find the number of cubic feet of masonry in a bridge arch of the following dimensions: span, 50 ft.; rise, one-fourth the span; thickness of masonry, 3 ft.; length of arch, 36 ft.

246. Find the weight of an iron dumb-bell consisting of two spheres of $4\frac{1}{2}$ in. diameter, joined by a cylindrical bar 6 in. long and 2 in. in diameter, an iron ball 4 in. in diameter weighing 9 lbs.

247. A zone of a sphere is 4 in. in thickness, the diameter of the base is 12 in., and that of the top 9 in.: find the convex surface and the volume.

248. The ends of a prismoid are rectangles, the corresponding dimensions of which are 18 ft. by 10 ft. and 12 ft. by 16 ft.; the height of the prismoid is 9 ft.; a section is made by a plane parallel to the ends at the distance of 3 ft. from larger end: show that the section is a square.

249. There is a cup in the shape of a conic frustum, 5 in. deep, the top diameter being 4 in., and the bottom diameter 3 in.: if it is filled with liquor, and three persons drink it successively in equal portions, what will be the depth of each draught?

250. A square-threaded screw with double thread is formed upon a cylinder 3 in. in diameter; the thread projects from the cylinder $\frac{5}{8}$ of an inch, and the screw rises 3 in. in four turns: find the volume if the screw be 9 in. in length.

251. A sphere 16 in. in diameter is divided into four parts of equal height by three parallel planes: find the volume of each part.

252. A well is to be constructed of the following dimensions: exterior diameter, 10 ft.; interior diameter, 7 ft.; height of cylinder, 30 ft. The cylinder rises 2 ft. above the surface of the ground, and is surrounded by a masonry top or platform 2 ft. in width (beyond the cylinder all round), and 5 ft. in depth, being 3 ft. below the surface of the ground, and 2 ft. above it. ($\pi = 3{\cdot}14159$.) Calculate (1) the quantity of masonry in the cylinder, and (2) the quantity in the platform.

253. A conical glass, whose depth is 4 in., and width at top 6 in., is filled with water: if a sphere of 6-in. diameter be placed in the glass, how many square inches of its surface will be immersed?

254. Find the edge of the greatest cube that can be cut out of a cone whose vertical angle is 60° and height 10 in.

255. What will be the cost of arching over a room 32 ft. long and 20 ft. span, the arch being segmental, with a rise of $\frac{1}{8}$ of span, and thickness of 9 in., the cost of masonry being Rs. 35 per 100 cub. ft.?

256. A pint tankard is in the form of a frustum of a circular cone; its height is $4\frac{1}{2}$ in., and the diameter of its base $3\frac{1}{2}$ in., both measurements taken inside: find the diameter of the top. ($\pi = 3{\cdot}1416$.)

257. Find the number of cubic yards of earth in a portion of a railway cutting 12 chains in length: the following numbers representing the areas in square yards of a series of transverse sections taken at intervals of 1 chain: 290, 264, 276, 268, 280, 274, 254, 250, 268, 240, 232, 226, 220.

258. Each edge of a regular tetrahedron measures 4 in.: find the volume.

259. Find the number of square miles between the 30th and 45th parallels of latitude, assuming the earth's radius to be 4000 miles. ($\pi = 3{\cdot}1416$.)

260. Prove that the area of a regular polygon of an even number of sides inscribed in a circle is a mean proportional between the areas of the inscribed and circumscribed polygons of half the number of sides.

261. A tin funnel consists of two parts; one part is conical, the slant height being 6 in. and the end circumferences 20 in. and $1\frac{1}{4}$ in. respectively, the other part is a cylinder 8 in. long and $1\frac{1}{4}$ in. in circumference: find the number of square inches of tin.

262. A frustum of a square pyramid has the area of the base four times

that of the top: show that its volume is seven-twelfths of that of a prism of equal base and altitude.

263. A cylindrical glass, height 8 in., diameter 3 in., is filled to a depth of 5 in. with water; the glass is then tilted till the water is about to run out: find the area of the surface of the water.

264. The height of a frustum of a pyramid is 4 in., the lower end is a rectangle which is 9 in. × 12 in., the upper end is a rectangle of which the longer side is 8 in.: find the volume of the pyramid.

265. What will be the expense of excavating the foundation for a house 50 ft. × 30 ft., to be erected on a piece of ground sloping to the south uniformly at the rate of one in 62? The front, of 50 ft., looking due south, and thus agreeing with the horizontal direction of the ground, is to be dug everywhere to the depth of 10 ft., and at that depth the ground-floor is to be carried on a level surface to the back, the excavation being thus deeper at the back than at the front. It is to be excavated, and the earth removed, at the rate of 10*d.* per cubic yard.

266. Three labourers are required to erect a conical mound of earth, and each is to perform the same share in the work, and to commence when his predecessor has completed his task: if the altitude of the cone is to be 20 ft., find the altitude of the portion contributed by each labourer.

267. A bucket is filled twenty-seven times with water from a circular well. It is found that the water fills 1 ft. $2\frac{1}{4}$ in. The bucket consists of a frustum of a cone whose height is 10 in., and the diameters of whose ends are 9 in. and 12 in. Find diameter of the well.

268. What is the length of the edge of the largest cube that can be cut out of a right cone of the following dimensions: diameter of base, 12 in.; height, 18 in.? *Note.*—The base of the cube is to be in the base of the cone.

269. A flower-bed is to be made on the space common to four equal circles whose centres are the angular points of a square of area 69 sq. ft. 64 sq. in., and radii equal in length to a side of the square: find the area of the flower-bed.

270. Each edge of a pyramid on a triangular base is equal to $12\sqrt{3}$ ft.: find the diameter of the greatest cylinder that can be cut out of it, the height of the cylinder being equal to its diameter.

271. Find to the nearest cubic foot the quantity of masonry in a bridge arch of 30 ft. span., $7\frac{1}{2}$ ft. rise, 3 ft. thick, and 27 ft. wide, and cost of constructing same at Rs.35 per 100 cub. ft.

272. Divide a cone into five equal parts by sections parallel to the base, and find the altitude of each part, the height of the cone being 20 in.

273. A cubical box with a lid is made of planking, and weighs 10 lbs.; its interior diagonal is 3 ft.: find its thickness, if 1 cub. ft. of the planking weigh 40 lbs.

274. Find the number of balls, each 6 in. in diameter, in a triangular pile consisting of twenty courses, top course being a single ball; and also the height of the pile.

275. The base of a cone is a circle of 12 in. diameter, and its height is also 12 in. A slice is cut from the one side of the cone by a plane passing through the vertex and cutting the base at the distance of 3 in. from the centre, but not including the centre: find the solid content of the slice.

276. A quadrilateral in a circle is bisected by one of its diagonals and trisected by the other. The two adjacent sides, which contain an obtuse angle and are subtended by the trisecting diagonal, are a and c respectively. Show that the area $= \frac{1}{4}\sqrt{\{34a^2c^2-(a^4+c^4)\}}$.

277. How many square yards of canvas are required to make a conical tent 9 ft. high, such that a man of 6 ft. could stand anywhere inside, within a radius of 2 ft. from the centre without stooping? ($\pi = 3{\cdot}1416$.)

278. The sides of a circular reservoir are inclined at an angle of 30° to the horizon, and the diameter of the horizontal bottom is 60 ft. : find the number of gallons contained in it when the water is 10 ft. deep.

279. The arch masonry of a semicircular arched bridge of 40 ft. span and 25 ft. wide increases in thickness from the crown to the springing as follows : along the first 10 ft. of the curve on each side of the crown the thickness is 1 ft. 3 in. ; along the second 10 ft. on each side, 1 ft. 6 in. ; and so on, increasing 3 in. on passing through every successive 10 ft. down to the springing line. Calculate the cost at Rs.28 per 100 cub. ft. ($\pi = 3{\cdot}1416$.)

280. The base of a prismoidal solid is a square, and the top a regular octagon, four alternate sides of which are parallel to the sides of the base. The altitude of the solid is 6 ft., the sides of the base 3·5 ft., and those of the top 1 ft. Find its volume.

281. A drinking water-trough for cattle is 13 ft. long and 3 ft. wide at top, 10 ft. long and 1 ft. wide at bottom, and 1 ft. deep, and the sides and ends slope symmetrically : how many gallons of water does it hold when the water is 7 in. deep ?

282. A wire cable is formed of six wires twisted round a central one, the diameter of each being $\frac{1}{8}$ in. ; the central wire is straight, and the others make one turn in 8 in. : find the volume of a yard of cable.

283. Four spheres are piled in a heap, three on the base and one on the top ; the diameter of each is 10 ft. : find the vertical height of the pile.

284. A perfectly flexible rope of 4 in. in diameter is coiled closely on the ground, and there are twelve complete coils : find the length of the rope in feet.

285. Brass wire $\frac{1}{8}$ in. thick and of circular section weighs 4 ozs. per foot : find weight of a length of wire sufficient to make twenty-five complete turns round a cylindrical shaft 3 ft. in diameter.

286. A road having a gradient of 1 in 20 is cut through the ridge of a hill, the sides of which have a slope of 1 in ten ; the length, measured from its entrance at one side to its exit at the other, is 500 ft., width 30 ft., and sides perpendicular : find the quantity of cutting in the same.

287. A spherical sector is removed from a solid sphere such that the spheric surface removed is one-sixteenth of the surface of the sphere : find what fraction of the sphere has been removed.

288. A rod whose section is an equilateral triangle of side a is bent so as to form a circular hoop of internal radius r ; one face of the rod is perpendicular to the plane of the hoop : find the ratio of the volumes of the two hoops that could be thus formed.

289. A conical wineglass, of height h and radius of base a, is held with axis vertical. Water is poured to such a depth that on inserting a heavy sphere of radius r the water completely fills the space between the sphere and the cone. If the sphere is wholly submerged, find the depth of water necessary.

290. The shaft of an obelisk is formed from a prism on a square base by truncating the edges so that the top is reduced to a regular octagon inscribed in the originally square top, and the bottom is unaltered : find its volume. (Height $= h$, side of square $= a$.)

291. Prove the formula $V = \frac{d}{3}\{A_1 + A_{2n+1} + 2(A_3 + A_5 + \ldots + A_{2n-1}) + 4(A_2 + A_4 + \ldots + A_{2n})\}$ for determining approximately the volume of earth in an embankment, and explain the symbols used.

292. The barrel of a shot-gun consists of a frustum of a cone, total diameter at breech 0·836 in., at muzzle 0·764 in. It is hollowed out to a diameter of 0·62 in. throughout. Find the weight of copper in the barrel, supposing that a solid rod of the same external dimensions as the barrel would

have weighed 2 lb. 6 ozs., and that there is 0·12 per cent. of copper in the composition of the metal.

293. A canal lock, with two flood gates at one end, is filled to 8 ft. above its original level in one and a half minutes: supposing the lock to be 176 ft. long and 12 ft. broad, and the water to flow in uniformly at the rate of 4 miles an hour, find the superficial area of the two flood gates.

294. I want a roller 4 ft. in length, and to weigh 8 cwt. It is to be made of freestone of the specific gravity of 2½. What must be its diameter?

295. A conical glass, of depth 4 in. and breadth at top 3 in., is filled with water: if a glass rod ½ in. in diameter be pushed into it as far as it will go and held upright, how much water will overflow?

296. The diameters of the ends of a frustum of a cone are respectively 20 ft. and 16 ft., and the height of the frustum is 5 ft.; the frustum is divided into three equal parts by planes parallel to the ends; find the distances of the planes from the smaller end.

297. A square-based pyramidal monolith of granite contains as many solid feet as a square, having its side equal to the perpendicular height would embrace superficial ones, the sides of the base of the pyramid itself being equal to one-half the slant edge: find dimensions and cost of polishing at 10 annas per square foot.

298. A spiral spring consists of nine complete coils; its vertical height is 17 in., the diameter of its transverse section is 1 in., and the mean radius of the spiral 8 in., and the solid is bounded by two horizontal planes: find its volume.

299. A field is in the form of an isosceles triangle, and measures 200 yds. along each of its equal sides, and 240 yds. along the base: what must be the length of a tether fixed at its apex and to a horse's nose to enable him to graze exactly one-sixth of it?

300. There is a vessel in the form of a frustum of a cone standing on its smaller end, whose volume is 8·67 cub. ft. and depth 21 in., and the diameter of its top is to that of its base as 7 : 5. A globe, the number representing whose volume is two and a half times that representing its surface, is put into it. Show that the diameters of the vessel are 35 and 25 in. approximately, and that of the globe 15 in.; and find also the volume of water which would be required just to cover the globe.

301. On ground having a uniform slope of 6 horizontal to 1 vertical an earthen mound is to be constructed. The top to be horizontal, and in the form of a square $ABCD$ of 18 ft. side; the corners A and B each to be 7 ft., and the corners C and D each 10 ft., vertically above the original ground surface. The sides of the mound to have a slope of 1 horizontal to 1 vertical (45°). Find in cubic feet the volume of earth required for the construction.

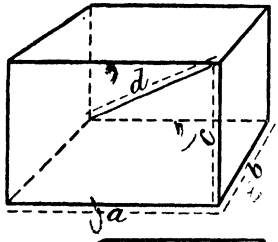
302. If the inscribed circle of a square of 1 ft. side be removed, and the remaining figure be made to rotate about one of the diagonals, find the volume of the solid thus generated. Express the answer in cubic feet to three places of decimals.

303. Two spheres of 3 in. and 1 in. diameter stand on a horizontal plane in such a position that a vertical line passes through the centres of both. A hollow conical vessel touching both spheres also stands on the horizontal plane. Find the volume of air contained in the hollow cone under the above conditions.

CHAPTER XXXVIII.

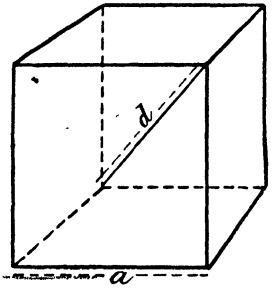
COLLECTION OF FORMULÆ—VOLUMES AND SURFACES OF SOLIDS.

203. Rectangular solids.



(i.) $V = abc$
(ii.) $V = A_1c = A_2b = A_3a$
(iii.) $V = \sqrt{A_1A_2A_3}$
(iv.) $S = 2(ab + bc + ca)$
(v.) $d = \sqrt{a^2 + b^2 + c^2}$

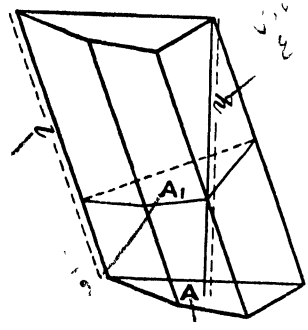
where V = volume; S = whole surface; a = length; b = breadth; c = depth; A_1 = area of base; A_2 = area of side; A_3 = area of end; d = diagonal.



Cubes.

(i.) $V = a^3$
(ii.) $S = 6a^2$
(iii.) $d = a\sqrt{3}$

where V = volume; S = whole surface; a = edge; d = diagonal.



Prisms and cylinders.

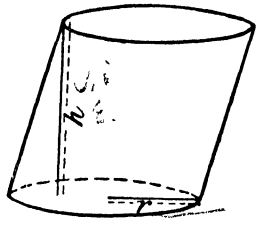
(i.) $V = Ah$
(ii.) $V = A_1l$
(iii.) $S = pl + 2A$

where V = volume; S = whole surface; A = area of base; A_1 = area of cross-section; h = height; l = length; p = perimeter of cross-section.

Circular cylinders.

$$V = \pi r^2 . h$$

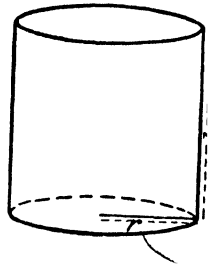
where V = volume; r = radius of base; h = height.



Right circular cylinders.

(i.) $V = \pi r^2 h$
(ii.) $S = 2\pi r(h + r)$

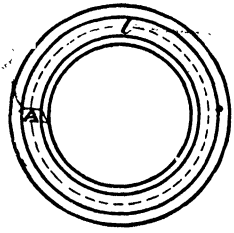
where V = volume; S = whole surface; r = radius of base; h = height.



Rings.

(i.) $V = Al$
(ii.) $S = pl$

where V = volume; S = whole surface; A = area of cross-section; l = length or mean circumference; p = perimeter of cross-section.



Cylindrical rings.

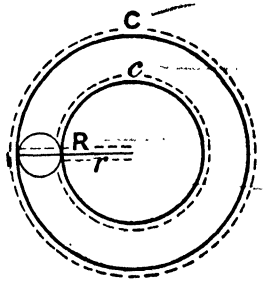
(i.) $V = \frac{\pi^2}{4}(R + r)(R - r)^2$

(ii.) $V = \frac{1}{32\pi}(C + c)(C - c)^2$

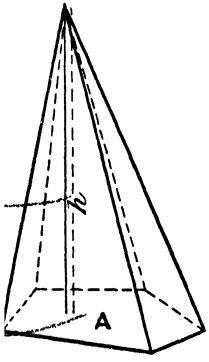
(iii.) $S = \pi^2(R^2 - r^2)$

(iv.) $S = \frac{1}{4}(C^2 - c^2)$

where V = volume; S = whole surface; R = outer radius; r = inner radius; C = outer circumference; c = inner circumference.

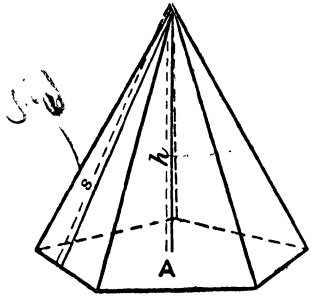


Pyramids and cones.



$$V = \tfrac{1}{3}Ah$$

where V = volume; A = area of base; h = height.

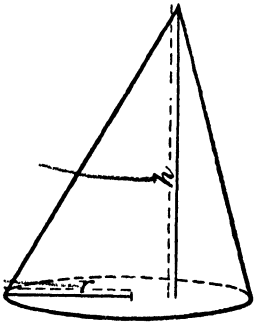


Right regular pyramids.

(i.) $V = \frac{1}{3}Ah$
(ii.) $S = \frac{1}{2} \cdot ps + A$

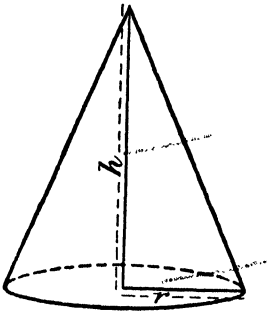
where V = volume; S = whole surface; A = area of base; p = perimeter of base; s = slant height.

Circular cones.



$$V = \tfrac{1}{3} \cdot \pi r^2 \cdot h$$

where V = volume; r = radius of base; h = height.

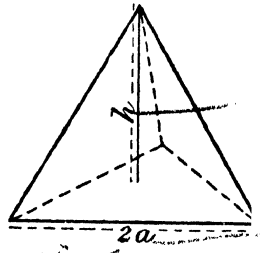


Right circular cones.

(i.) $V = \frac{1}{3}\pi r^2 h$
(ii.) $S = \pi r(\sqrt{h^2 + r^2} + r)$

where V = volume; S = whole surface; h = height; r = radius of base.

Regular tetrahedrons.



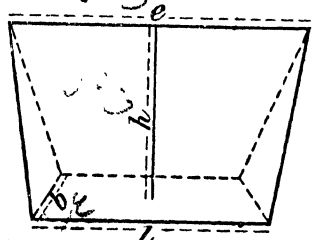
$$\text{(i.)}\quad V = \frac{2\sqrt{2}}{3}a^3$$

$$\text{(ii.)}\quad S = 4a^2\sqrt{3}$$

$$\text{(iii.)}\quad h = 2a\sqrt{\tfrac{2}{3}}$$

where V = volume; S = whole surface; $2a$ = edge; h = height.

Wedges on rectangular bases.

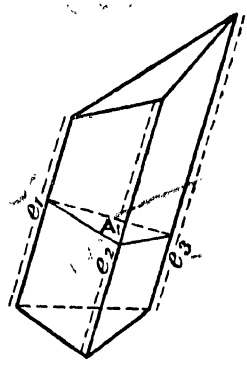


$$\text{(i.)}\quad V = \frac{bh}{6}(2l + e)$$

$$\text{(ii.)}\quad V = \frac{A_1}{3}(2l + e)$$

where V = volume; l = length of base; b = breadth of base; e = edge; A_1 = area of cross-section.

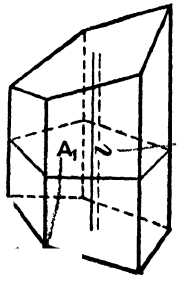
Wedges on trapezoidal bases, or oblique frusta of triangular prisms.



$$V = A_1 \cdot \frac{e_1 + e_2 + e_3}{3}$$

where V = volume; A_1 = area of cross-section; e_1, e_2, e_3 are the lengths of the three parallel edges.

Oblique frusta of any right regular prisms.



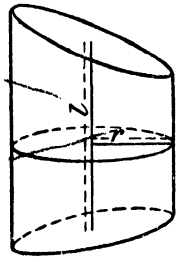
$$\text{(i.)}\quad V = A_1 l$$

$$\text{(ii.)}\quad S = pl$$

where V = volume; S = lateral surface; A_1 = area of cross-section; l = mean length; p = perimeter of cross-section.

Note.—By *mean* length is meant the average length of the parallel edges, that is—

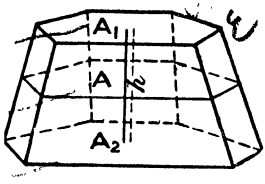
$$\frac{\text{sum of parallel edges}}{\text{number of parallel edges}}$$



Oblique frusta of right circular cylinders.

(i.) $V = \pi r^2 . l$
(ii.) $S = 2\pi r . l$

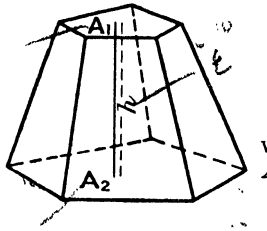
where V = volume; S = curved surface; r = radius of cross-section; l = mean length.



Prismoids.

$$V = \frac{h}{6}(A_1 + A_2 + 4A)$$

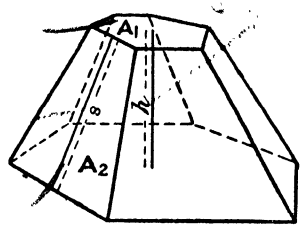
where V = volume; h = height; A_1 and A_2 are the areas of the ends; A = area of mid-section parallel to the ends.



Frusta of pyramids and cones.

$$V = \frac{h}{3}(A_1 + A_2 + \sqrt{A_1 A_2})$$

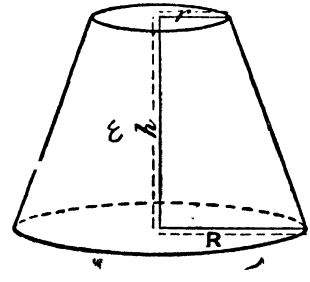
where V = volume; h = height; A_1 and A_2 are the areas of the ends.



Frusta of right regular pyramids.

(i.) $V = \frac{h}{3}(A_1 + A_2 + \sqrt{A_1 A_2})$
(ii.) $S = \frac{1}{2}s . (P + p)$

where V = volume; S = lateral surface; h = height; A_1 and A_2 are the areas of the ends; P and p are the perimeters of the ends; s = slant height.

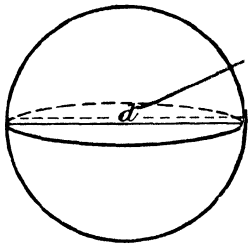


Frusta of right circular cones.

(i.) $V = \frac{\pi h}{3}(R^2 + r^2 + Rr)$
(ii.) $S = \frac{1}{2}s(C + c)$
(iii.) $S = \pi s(R + r)$

where V = volume; S = curved surface; R and r are the radii of the ends; C and c are the circumferences of the ends; s = slant height.

Spheres.



(i.) $V = \frac{\pi d^3}{6}$

(ii.) $V = \frac{4}{3}\pi r^3$

(iii.) $S = \pi d^2$

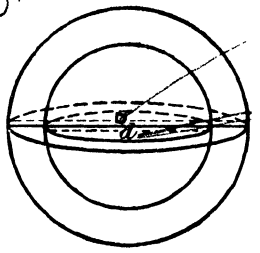
(iv.) $S = 4\pi r^2$

where V = volume; S = surface; d = diameter; r = radius.

Spherical shells.

(i.) $V = \frac{\pi}{6}(D^3 - d^3)$

(ii.) $V = \frac{4\pi}{3}(R^3 - r^3)$



and when the thickness of the shell is very small compared with the outer diameter—

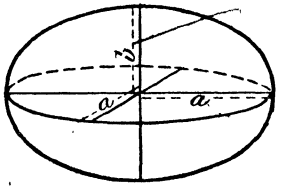
(iii.) $V = \pi \cdot D^2 \cdot h$ nearly

also, when the thickness of the shell is nearly equal to the outer radius

(iv.) $V = \frac{\pi}{3} \cdot D^2 \cdot h$ nearly

where V = volume; D = outer diameter; d = inner diameter R = outer radius; r = inner radius; h = thickness.

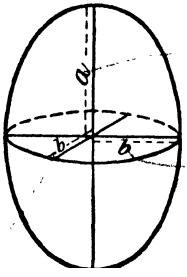
Oblate spheroids.



$V = \frac{4}{3}\pi a^2 b$

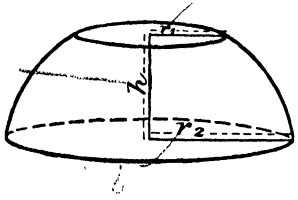
where V = volume; a = semi-major axis; b = semi-minor axis.

Prolate spheroids.



$V = \frac{4}{3}\pi a b^2$

where V = volume; a = semi-major axis; b = semi-minor axis.

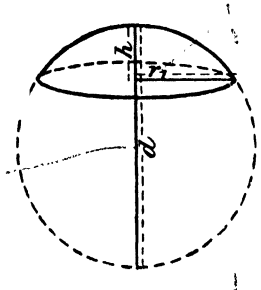


Zones of spheres.

(i.) $V = \frac{\pi h}{6}\{3(r_1^2 + r_2^2) + h^2\}$

(ii.) $S = \pi dh$

where V = volume; S = curved surface; r_1, r_2 are the radii of the two ends; h = height; d = diameter of the sphere.



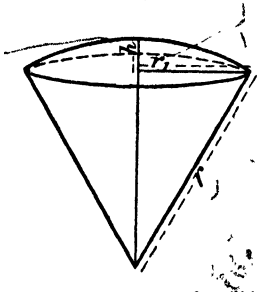
Segments of spheres.

(i.) $V = \frac{\pi h}{6}(3r_1^2 + h^2)$

(ii.) $V = \frac{\pi h^2}{6}(3d - 2h)$

(iii.) $S = \pi dh$

where V = volume; S = curved surface; r_1 = radius of the base of the segment; h = height; d = diameter of the sphere.



Sectors of spheres.

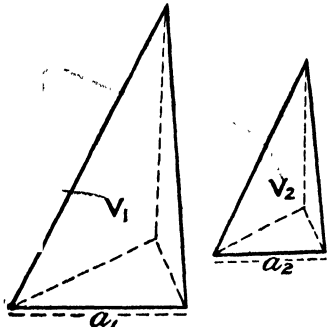
(i.) $V = \frac{2}{3}\pi r^2 . h$

(ii.) $V = \frac{1}{3} . rs$

(iii.) $S = \pi r\{2h + \sqrt{(2rh - h^2)}\}$

where V = volume; S = whole surface; r = radius of the sphere; h and s are the height and curved surface of the segment of the sphere that forms the base of the sector.

Note.—Formula (ii.) follows from formula (i.), since $s = 2\pi r . h$ (§ 198).



Similar solids.

(i.) $V_1 : V_2 = a_1^3 : a_2^3$

(ii.) $S_1 : S_2 = a_1^2 : a_2^2$

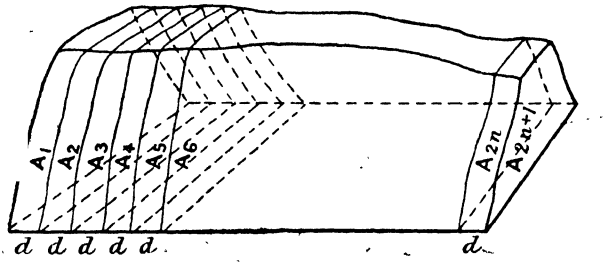
(iii.) $a_1 : a_2 = \sqrt[3]{V_1} : \sqrt[3]{V_2}$

(iv.) $a_1 : a_2 = \sqrt{S_1} : \sqrt{S_2}$

where V_1 and V_2 are the volumes; S_1 and S_2 are the surfaces; a_1 and a_2 are corresponding linear dimensions of the first and second similar solids respectively.

Irregular solids whose opposite ends are plane figures lying in parallel planes.

$$V = \frac{d}{3}\{A_1 + A_{2n+1} + 2(A_3 + A_5 + \ldots + A_{2n-1}) + 4(A_2 + A_4 + \ldots + A_{2n})\}$$



where V = volume; $2n$ = number of equal parts into which the length of the solid is divided by planes parallel to its ends; d = common distance between the parallel planes; A_1, A_2, A_3 . . . A_{2n}, A_{2n+1} are the areas of the transverse sections of the figure made by the parallel planes taken in order.

Tables.

No.	Square.	Cube.	Square root.	Cube root.
1	1	1	1·00000	1·0000
2	4	8	1·4142135	1·2599
3	9	27	1·7320508	1·4422
4	16	64	2·00000	1·5874
5	25	125	2·23607	1·7099
6	36	216	2·44949	1·8171
7	49	343	2·64575	1·9129
8	64	512	2·82843	2·0000
9	81	729	3·00000	2·0800
10	100	1000	3·16228	2·1544
11	121	1331	3·31662	2·2239
12	144	1728	3·46410	2·2894
13	169	2197	3·60555	2·3513
14	196	2744	3·74166	2·4101
15	225	3375	3·87298	2·4662
16	256	4096	4·00000	2·5198
17	289	4913	4·12311	2·5712
18	324	5832	4·24264	2·6207
19	361	6859	4·35890	2·6684
20	400	8000	4·47214	2·7144

$$\pi = 3{\cdot}1415926536$$
$$\sqrt{\pi} = 1{\cdot}7724538509$$
$$\pi^2 = 9{\cdot}8696044011$$
$$\frac{1}{\pi} = 0{\cdot}3183098862$$
$$\frac{1}{\pi^2} = 0{\cdot}1013211836$$

One cubic foot of pure water weighs 997·137 oz. (Av.) = 1000 ozs. nearly

One gallon measure contains 277·274 cub. in. = 0·16046 cub. ft.
= $277\frac{1}{4}$ cub. in. nearly

$$\frac{1}{277{\cdot}274} = 0{\cdot}0036065$$

ANSWERS

Examples—XIX.

1. 18,016 cub. in. 2. 125,648 cub. in. 3. 4 cub. yds. 25 cub. ft. 176 cub. in.
4. 2 cub. yds. 25 cub. ft. 80 cub. in. 5. 35·84 cub. ft. 6. 27,000 oz.
7. 69·3125 cub. in. 8. 500 lbs.

Examples—XX.

1. 210 cub. ft. 2. 111 cub. ft. 54 cub. in.
3. 7 cub. yds. 7 cub. ft. 1512 cub. in. 4. 30 ft. 2 in. 5. 5 ft. 2 in.
6. 9 in. 7. 9 sq. ft. 42 sq. in. 8. 1 yd. 1 ft. 3 in.
9. 2 sq. yds. 4 sq. ft. 132 sq. in. 10. 7 cub. yds. 16 cub. ft. 520 cub. in.
11. 1 yd. 2 ft. 10 in. 12. 6 cub. yds. 22 cub. ft. 16 cub. in.
13. Rs. 1960. 14. 10,240 bricks. 15. 2283 galls.
16. 876·5625 lbs. 17. 15 tons 12 cwt. 2 qrs. 18. 11 in.
19. 10 cub. yds. 1152 cub. in. 20. 9 cub. ft. 48 cub. in.
21. 22 cub. ft. 1288 cub. in. 22. 7 cub. yds. 1 cub. ft. 189 cub. in.
23. 1331 cub. yds. 24. $421\frac{7}{8}$ cub. yds. 25. 1 ft. 7 in.
26. 1 yd. 7 in. 27. 1 yd. 2 ft. 1 in. 28. 5 yds. 1 ft. 5 in.
29. 4 ft. 5 in. 30. 24·06 in. 31. 345,600,000 sq. in.
32. 5 cub. ft. 960 cub. in. 33. 1714 cub. in. 34. 1530 cub. in.
35. 4·3 . . . in. 37. 12 min. 53 secs. 38. 1 ft. 2 in.; 2 ft. 11 in.; 4 ft. 1 in.
39. 3 cub. ft. 12 cub. in. nearly. 40. 58·4 in. nearly.
41. 45 in. nearly. 42. 4 ft. 43. 12 ft.; 9 ft. 44. 1 ft. 3 in.

Examination Questions—XX.

1. Yes; because edge of cube = 15·9 . . . ft.; side of square = 16·2 ft.
3. 42·4263 ft. 4. 6 days nearly. 5. 8242·408 cub. in.
6. 9 cub. ft.; 3182 cub. in. 7. $\frac{1}{2}\frac{9}{6}$ oz.; $1\frac{7}{16}$ oz. 8. $314\frac{1}{2}$ cub. ft.
9. 5196·15 cub. in. 10. $17\frac{1}{2}$ ft. broad; 13 ft. high.
11. $31,516\frac{4}{9}$ bricks. 12. 12,288 bricks. 13. 60 in.
14. $88,392\frac{6}{7}$ tons. 15. 0·000046 in. 16. 2·75 in. nearly.
17. 279 ft.; 93 ft.; 124 ft. 18. £1 os. 8d. 19. 13·61 ft.; 46·03 sq. ft.

Examples—XXI.

1. 15 cub. ft. 10′. 2. 41 cub. ft. 11′ 1″ 3‴. 3. 29 cub. ft. 5′ 1″ 8^{iv}. 11^{iv}.
4. 42 cub. ft. 11″ 6‴ 7ᵛ. 5. 14 cub. ft. 8′ 7″ 2‴ 11^{iv} 9^{v}.
6. 1 cub. prime 9 cub. secs.

Examples—XXII.

1. 7 cub. ft. 966 cub. in. **2.** 1 cub. yd. 5 cub. ft. 396 cub. in.
3. 5 cub. yds. 8 cub. ft. 1368 cub. in. **4.** 5 cub. yds. 10 cub. ft. 768 cub. in.
5. 1 ft. 6 in. **6.** 2 ft. 9 in. **7.** 1 sq. yd. 1 sq. ft. 72 sq. in.
8. 2 sq. yds. 2 sq. ft. 84 sq. in. **9.** 1134 cub. in. **10.** 450 cub. in.
11. 1620 cub. ft. **12.** 1 ft. 3 in. **13.** 2 yds. 2 ft. 4 in.
14. 1 ton 14 cwt. 3 qrs. 5 lbs. 1 oz. nearly. **15.** $53\frac{3}{7}$ cub. ft.
16. $26\frac{39}{56}$ cub. ft. **17.** 24 cub. yds. 23 cub. ft. 594 cub. in.
18. 7·2 . . . in. **19.** 15·6 . . . in. **20.** 16·5 . . . in.
21. 17,670 galls. **22.** 404·0883 cub. ft. **23.** 1087 coins.
24. Rs.47 10 annas 8 pies. **25.** 24,446 yds. nearly. **26.** 12·217 cub. ft.
27. Rs.68 15 annas $1\frac{1}{2}$ pies. **28.** 44·6 in. nearly. **29.** 3·6 cub. ft. nearly.
30. 11·15 in. ; 13·15 in. **31.** 10 ft. 10 in. nearly. **32.** 957·6 galls.
33. $536,604\frac{1}{2}$ cub. yds. **34.** 9 tons 9 cwt. 1 qr. 9 lbs. 12·8 ozs.
35. 0·72 . . . in. **36.** 4·08 ft. nearly. **37.** 14·76 cub. ft.
38. Rs.327 6 annas $1\frac{1}{2}$ pies. **39.** 525 cub. ft. **40.** 8·72 cub. in.
41. 39·13 cub. in. **42.** 123·29 cub. in. **43.** 220·98 cub. in.
44. 554·26 cub. in. **45.** 9·76 cub. in. **46.** 8·68 cub. in.
47. 6 in. **48.** 4 ft. 4 in.

Examination Questions—XXII.

1. 0·416 in. **2.** 1·8638 tons. **3.** 974·278 . . . cub. ft.
5. $2862\frac{1}{4}$ cub. ft. **6.** $3033\frac{9}{14}$ cub. ft. **7.** 2·598 cub. ft.
8. 1848 cub. ft. **9.** $35,982\frac{3}{5}$ cub. yds. ; $5\frac{1}{3}$ ft. **10.** 2595 cub. ft.
11. 14,627 cub. ft. nearly. **12.** 2178 tons. **13.** 1697 cub. ft. nearly.
14. $153\frac{3}{14}$ cub. ft. **15.** 338 cub. ft. nearly ; Rs.118 nearly.
16. 107 ft. **17.** 3374·9 cub. ft. ; Rs.1012, 7 annas 6 pies.
18. 2362·4 cub. ft. nearly ; Rs.708 11 annas 8 pies nearly. **19.** 33,750 cub. ft.
20. 1104 cub. ft. nearly. **21.** 1500 cub. ft. **22.** 157,023 acres.
23. $18,327\frac{3}{11}$ ft. **24.** $1643\frac{25}{112}$ cub. ft. ; $670\frac{101}{112}$ cub. ft.
25. $1195\frac{1}{14}$ cub. ft. ; $487\frac{13}{14}$ cub. ft. **26.** 5·9 in.
28. $1270\frac{25}{28}$ cub. ft. **29.** 373·395 lbs. **30.** 90 cub. ft.
31. 489 nearly. **32.** $1257\frac{1}{7}$ cub. ft. ; 550 cub. ft.
33. 1·276 . . . in. **34.** $97,745\frac{5}{11}$ yds. **35.** $339\frac{3}{7}$ lbs.
36. 0·00554 sq. in. **37.** 5·4 galls. nearly. **38.** 1·36 ft. nearly.
39. 651·9 lbs. **40.** 3·5625 galls. **41.** $1018\frac{3}{7}$ cub. in.
42. $\frac{7}{16}$ in. **43.** $2710\frac{3}{7}$ cub. ft. ; Rs.677 10 annas $10\frac{3}{7}$ pies.
44. 0·207 in. nearly. **45.** 22·135 in. **46.** Rs.497 6 annas $4\frac{4}{33}$ pies.
47. 1·4 in. nearly. **48.** 2534·3 cub. in. **49.** 10·283 . . . cub. in.
50. 6196·773 lbs. **51.** 8200·83 . . . **52.** 3·286 . . . cub. in. ; 20·92 ozs.
54. $2\frac{1}{3}$ in. **55.** $63\frac{7}{11}$ in. **56.** Rs.5091 6 annas $10\frac{2}{7}$ pies.
57. $127,285\frac{5}{7}$ cub. ft. **58.** 3·841 in.

Examples—XXIII.

1. 4 cub. ft. 1028 cub. in. **2.** 14 cub. ft. 1509 cub. in.
3. 40 cub. ft. 292 cub. in. **4.** 3 cub. yds. 24 cub. ft. 668 cub. in.
5. 10 ft. **6.** 6 yds. 3 in. **7.** 11 sq. ft. 36 sq. in.
8. 7 sq. ft. 78 sq. in. **9.** $513\frac{1}{3}$ cub. in. **10.** 53 cub. ft. 816 cub. in.
11. 59 cub. ft. 57·6 cub. in. **12.** 3 cub. yds. 25 cub. ft. 1632 cub. in.
13. 3 in. **14.** 3 in. **15.** 7·74 in. nearly.
16. 2 ft. 6 in. **17.** 1020 cub. ft. **18.** 364 cub. yds.
19. 336 cub. ft. **20.** 1 cub. ft. 1352 cub. in. **21.** 0·5773 cub. ft.

22. $94\frac{3}{4}$ cub. ft. **23.** 1885 cub. in. nearly. **24.** 114 lbs.
25. $1178\frac{1}{4}$ cub. in. **26.** 34,992 cub. ft. **27.** 32·725 cub. in.
28. 3·844 in. **29.** 19 cub. yds. 6 cub. ft. 768 cub. in. **30.** 8·973 cub. in.

Examination Questions—XXIII.

1. 935·307 cub. ft. **2.** $\frac{a^3}{6}$. **3.** 50·911 cub. ft.
5. 81·1898 . . . cub. ft. **8.** 1173·4 . . . cub. ft.
9. 1885·618 . . . cub. ft. **10.** $288{,}000\sqrt{2}$ cub. ft.
11. 36,373 cub. ft. nearly. **12.** $166\frac{2}{3}$ cub. ft. **13.** 203·6467 . . . cub. in.
14. 2708 cub. in. **16.** $\frac{2}{3}abh$; $a : 2\sqrt{\frac{1}{4}a^2 + \frac{2}{3}b^2}$.
17. 311·769 cub. ft. **18.** 275·2 . . . cub. ft.
19. 6363·96 cub. ft. **20.** 3·399 cub. ft.
21. $1493\frac{1}{18}$ cub. ft.; 801·249 . . . tons; £104,696,606 8*s*.
23. $64\frac{1}{6}$ cub. ft. **24.** 391 93 . . . cub. ft. **25.** $116\frac{27}{154}$ cub. ft.
27. $37\frac{2}{7}$ cub. in. **28.** $339\frac{2}{7}$ cub. in. **29.** $30\frac{6}{35}$ cub. in.
30. 6268 galls. nearly. **31.** $70\frac{5}{7}$ cub. ft. **32.** 2·978 in.
33. 664 cub. ft. nearly. **34.** $18\frac{3}{4}$ ft.; 19·40229 ft.; 10·0904 ft.
35. 331 cub. in. nearly. **36.** 3258·514 cub. in.
37. 63·39 cub. in. **38.** 1930·971 cub. in.

Examples—XXIV.

1. 1008 cub. in. **2.** 2 cub. ft. 1269 cub. in.
3. 1 cub. ft. 1536 cub. in. **4.** 554·25 cub. in.
5. 1 cub. ft. 414 cub. in. **6.** 114·890625 tons. **7.** 15·588 cub. ft.

Examination Questions—XXIV.

1. $15\frac{1}{8}$ cub. in. **2.** 4 cub. ft. 228 cub. in. **3.** $30710\frac{1}{32}$ cub. metres.
5. 1155 cub. in.; 462 cub. in. **6.** $11\frac{79}{198}$ in. **7.** 4000 cub. in.
8. 59 cub. in.; 161 cub. in.; 239 cub. in.
9 35 cub. ft. **10.** 995·92 cub. in.
11. 360 cub. in.; 364 cub. in.; 1004 cub. in. **12.** 96·9948 cub. in.

Examples—XXV.

1. 93·5307 cub. ft. **2.** 33·798 cub. ft. **3.** 313·4 . . . cub. in.
4. 30·909 cub. ft. **5.** 112·94 cub. ft. **6.** 18·005 cub. ft.

Examination Questions—XXV.

1. 0·6486 . . . of its volume. **2.** 9933·658 . . . lbs.
3. 6π cub. ft. **4.** 0·277 cub. in.

Examples—XXVI. A.

1. 595,000 cub. ft. **2.** 55·35 cub. ft. **3.** 41·14 tons.
4. 14,666·6 cub. yds. **5.** 466 cub. in.; 394 cub. in. **6.** 294·448 cub. in.
7. 4736 galls. nearly. **8.** 33,650 cub. ft. **9.** $56{,}877\frac{1}{3}$ cub. yds

Examples—XXVI. B.

1. 19·324 cub. ft. **2.** 59·59 cub. in. **3.** 7 cub. ft. nearly.
4. 9·9 cub. ft. nearly. **5.** 9990 cub. in. **6.** 62·9 cub. ft.
7. 429 cub. ft. **8.** 57·5 cub. ft. **9.** 338·47 cub. ft.
10. 55·106 cub. yds. **11.** 1350·15 cub. in. **12.** 112·6 cub. in.; 84·6 cub. in.
13. 22,704 cub. in. **14.** 315·45 cub. ft.

Examination Questions—XXVI.

1. 296,050·4 galls.
2. £842 4*s.* 5½*d.*
3. 101,600 cub. ft.
4. 29,680 cub. in.
5. 12,580$\frac{20}{27}$ cub. yds.
6. 4366$\frac{2}{3}$ cub. ft.
7. 26,516$\frac{2}{3}$ cub. yds.
8. 12,466$\frac{2}{3}$ cub. yds.
9. 9·154 tons.
10. Prismoid, 36,120 cub. in.; wedge, 7800 cub. in.
11. 47,412$\frac{2}{3}$ cub. yds.
12. 647,333$\frac{1}{3}$ cub. ft.
13. 925,929$\frac{3}{28}$ cub. ft.
15. 1,020,000 cub. ft.
16. 198$\frac{2}{3}$ cub. ft.; 126$\frac{2}{3}$ cub. ft.
17. 202,400 cub. ft.
18. 73$\frac{11}{12}$ cub. ft.
19. Rs.16,159.
20. 6·9282 cub. ft.
21. 72 cub. ft. 651 cub. in.
22. 49$\frac{3}{5}$ cub. ft.
23. 38,026$\frac{2}{3}$ cub. yds.
24. 7600 cub. ft.
25. 648 cub. ft. nearly.
26. 13,416 cub. ft. nearly.
27. $\frac{nh}{12}\cot\frac{180°}{n}(a^2+b^2+ab)$.
29. 464,020 cub. ft. nearly.
30. 147 cub. ft.
31. 164·638 cub. ft.
32. Rs.125,077 5 annas 4 pies.
33. 136·15 cub. ft. nearly.
34. $912\sqrt{3}$ cub. ft.
35. 563·31 cub. in.
36. 57·62 galls.
37. 168$\frac{14}{27}$ cub. yds.
38. 38,217$\frac{1}{7}$ cub. in.
39. 12,370 cub. in. nearly.
40. 5 lbs. 3·9 ozs. (Av.) nearly.
41. 7372$\frac{5}{14}$ cub. ft.
43. 636 times nearly.
44. 2·05 in.
45. 4 cwt. 3 qrs. 22 lbs. 10·4 ozs.
46. 1 mi. 1005 yds. 1 ft. 3$\frac{3}{8}$ in.
47. 4826·25 cub. ft.
48. 917$\frac{7}{8}$ cub. ft.
51. 48 ft.
52. 4$\frac{3}{8}$ in.
53. 0·806 . . . in. per hour.
54. £30 2*s.* 2*d.* nearly.
55. 14·9 in. nearly; 24·9 in. nearly.
56. 91 galls. nearly.
57. Rs.958 9 annas 1$\frac{1}{4}$ pies.
58. Rs.14 8 annas 6$\frac{2}{7}$ pies.
60. 9654 cub. ft. nearly.
61. 141·8 cub. yds. nearly.
62. 94 lbs.
63. 68,954$\frac{2}{7}$ cub. ft.
64. 66 times.
65. 60,709½ cub. ft. nearly.
66. 2·94 in.

Examples—XXVII.

1. 381$\frac{6}{7}$ cub. ft.
2. 16·633 cub. ft.
3. 120·362 cub. in.
4. 3·376 cub. yds.
5. 7 ft.
6. 6 in.
7. 16·87 cub. in.
8. 370·40 cub. in.
9. 1909 bullets.
10. 104·4 galls. nearly.
11. 9 cwt. 2 qrs. 8$\frac{16}{21}$ lbs.
12. 0·344 mins.
13. 113·76 cub. ft.
14. 101·82 lbs.
15. 255·619 cub. in.
16. 18·0605 lbs.
17. 79·8809 lbs.
18. 9654$\frac{6}{7}$ cub. in.
19. 56·6931 cub. in.

Examination Questions—XXVII.

1. 31$\frac{3}{4}$ cub. ft.
2. 1·2599 . . . in.
3. 4, 6, 9.
5. 14$\frac{1}{7}$ cub. ft.
6. 4$\frac{1}{2}$ in.
7. 4851 cub. in.
8. 3,809,523 cub. mi. nearly
9. 2·53 in.
10. 4·06293 in.
11. $\frac{1}{3}$.
12. 1·2399 lbs.
13. 2 ft. 6 in.
14. 7$\frac{7}{8}$ lbs.
15. 14·1421 ft.
16. $\frac{1917601}{512000000}$ or $\frac{3}{800}$ nearly.
17. 16,036$\frac{4}{11}$ bullets.
18. 2 in.
19. π cub. ft.
20. 864 persons.
21. $\frac{2\pi}{3}$ cub. ft.
22. 165·873015 lbs.
23. 7·676 in.
24. 0·72204 in.
25. 201 lbs. 13$\frac{3}{5}$ oz.
26. Radius × 0·206.
27. 8π cub. ft.
28. 8·57 in.
29. 20·73 galls.
30. 77·78 cub. ft.
31. 1299·87 cub. in.
32. 196 lbs. nearly.
33. 168 lbs. 5¼ ozs.
34. $\frac{\pi}{3}$ cub. ft.
35. 20·08 lbs.

Examples—XXVIII.

1. $416\frac{20}{21}$ cub. in.
2. $1819\frac{1}{7}$ cub. in.
3. $32\frac{19}{21}$ cub. in.
4. $13,110\frac{3}{7}$ cub. in.
5. $113\frac{1}{7}$ cub. ft.
6. $1\frac{37}{81}$ cub. ft.
7. 792 cub. in. ; $2262\frac{8}{7}$ cub. in.
8. $135\frac{65}{81}$ cub. in. ; $252\frac{116}{567}$ cub. in. ; $135\frac{64}{81}$ cub. in.
9. 9·29 cub. in. nearly.
10. 17,880·76 cub. in.
11. 5 galls.
12. $81\frac{71}{84}$ cub. ft. ; $180\frac{5}{84}$ cub. ft. ; $180\frac{5}{84}$ cub. ft. ; $81\frac{71}{84}$ cub. ft.
13. 13 cub. ft. 636 cub. in.
14. $821\frac{1}{3}$ cub. in.

Examination Questions—XXVIII.

1. $360\frac{8}{21}$ cub. ft.
2. $\frac{5}{27}$.
3. $32\frac{23}{36}$ cub. ft.
4. $8\frac{8}{21}$ cub. ft.
5. 3797·619 cub. in.
6. 335·1 cub. in. ; 335·1 cub. in. ; 737·2 cub. in. ; 737·2 cub. in.
7. $1470\frac{8}{9}$ cub. in. ; 792 cub. in. ; 792 cub. in.
8. 370·47 cub. ft.
9. 40·88 lbs. nearly.
10. 12 ins.
11. 2076·904 cub. ft.
12. $2262\frac{8}{9}$ cub. ft. ; 792 cub. ft.
13. 18 galls.
14. 26·2 cub. in. nearly.
15. $458\frac{1}{3}$ cub. ft.
16. 0·375 cub. in.
17. 37·77 cub. ft.
18. 4·574 . . . in.
19. 6·299 ft.
20. 181·028 . . . lbs.
21. $188\frac{1}{7}$ cub. ft.

Examples—XXIX.

1. 343 : 8.
2. 1 ft. 3 in.
3. $79\frac{43}{108}$ cub. in.
4. 32·76 lbs.
5. 15 ft.
6. 15·874 in. ; 4·126 in.
7. 7 : 5.
8. 10 : 9.
9. $8\frac{1}{2}$ ft.
10. 7 : 8.
11. 2·38 . . . ft.
12. 8·32 ft. ; 2·16 ft. ; 1·51 ft.

Examination Questions—XXIX.

1. 7·98 in. nearly.
2. 0·271 of itself.
3. 1 : 7 : 19.
4. 16·2 . . . in.
5. 125 : 64.
6. 2·77 ft. nearly.
9. 1·9 . . . ft. ; 3·5 . . . ft.
10. 56·162 . . . in. ; 14·597 . . . in. ; 10·239 . . . in.
11. 58,320 lbs.
12. 3 : 5.
14. 13·8 . . . in. ; 3·6 . . . in. ; 2·5 . . . in.

Examples XXX.

1. $50\frac{1}{2}$ sq. ft.
2. 96 sq. ft. 124 sq. in.
3. 29 sq. yds. 8 sq. ft. 102 sq. in.
4. 77 sq. ft. 6 sq. in.
5. 20 sq. yds. 1 sq. ft. 72 sq. in.
6. 50 sq. yds. 96 sq. in.
7. 3*s.* $7\frac{31}{36}$*d.*
8. 7·57 ft. nearly.
9. 277 sq. ft. nearly.
10. 1000 sq. in.
11. 1200 sq. in.
12. 96 sq. ft. 100 sq. in.
13. 15·49 in.
14. £25 6*s.* 8*d.*
17. 4500 sq. ft.
18. 3360 sq. ft.
19. 92·784 sq. ft.
20. 89·656 sq. in.
21. 7·616 sq. ft.
22. £3 5*s.* $7\frac{1}{2}$*d.*
23. 10 in.
24. 1154 sq. in.
25. 1·4433 in.
26. 20 sq. ft.
27. 43·30127 sq. ft.
28. 619·80762 sq. in.
29. 1960 sq. in.
30. 1218 sq. in. nearly.
31. 12 in.
32. £1 10*s.* 4·9*d.*
33. $13,968\sqrt{3}$ sq. in.
34. 221·296 sq. in.
35. 2829 sq. in. nearly.
36. 9758 sq. in.
37. 1680 sq. in.
38. $115\frac{1}{2}$ sq. ft.
39. $33\frac{3}{4}$ sq. ft.
40. 6600 sq. ft.
41. 480 sq. in.
42. 209 sq. ft.
43. 61 sq. ft. 141 sq. in.
44. £173 12*s.* 3·2*d.*
45. 256 sq. in.

Examination Questions—XXX.

1. £37 16*s.* 4½*d.*
2. 2·8284 cub. ft.; 12 sq. ft.; 1·346 cub. ft.
3. 489 nearly.
4. £2 11*s.* 11·04*d.*
5. £1 10*s.* 10⅞*d.*
6. 12*s.* 3*d.*
7. 1 : $3\sqrt{2}$.
8. 39·687 ft.
9. 93·608 sq. ft.
10. 10·3 ft. nearly.
11. £10 10*s.*
13. 90 sq. ft.
14. $\frac{na}{2}\sqrt{\left\{\left(\frac{a^2}{4}\cot^2\frac{180^\circ}{n}\right)+h^2\right\}}$
15. Rs.25 8 annas 4 pies.
16. 173·20508 sq. ft.
17. £878,653 nearly.
18. 17·748 sq. ft.
19. 6·5 . . . sq. in.
20. Rs.160 nearly.
21. Rs.85 11 annas 2 pies.
22. 22·41 . . . sq. ft.
23. 125·6 sq. ft.
24. 435·099 sq. ft.
25. 1·113 sq. ft.

Examples—XXXI.

1. 66 sq. ft.
2. $50\frac{19}{28}$ sq. ft.
3. $8\frac{13}{72}$ sq. ft.
4. $2325\frac{5}{7}$ sq. in.
5. $14\frac{2}{3}$ sq. ft.
6. $54\frac{1}{13}$ sq. ft.
7. $10\frac{1}{2}$ in.
8. 7 in.
9. 1 ft. 3 in.
10. $3\frac{1}{2}$ in.
11. 7 in.
12. £4 6*s.* 9*d.*
13. 440 sq. in.
14. 2 : 1.
15. 153·92 sq. in.
16. 235·46 sq. in.
19. 396 sq. in.
20. $503\frac{37}{49}$ sq. in.
21. $79\frac{1}{49}$ sq. in.
22. $6\frac{1}{2}$ in.

Examination Questions—XXXI.

1. $579\frac{15}{56}$ sq. ft.
2. 1 in. or 2 in.
3. 4.
4. 1 : 2.
6. 14·357 in.
7. $19\frac{1}{11}$ in.

Examples—XXXII.

1. 53·16 sq. ft.
2. $77\frac{11}{14}$ sq. ft.
3. $8\frac{31}{36}$ sq. ft.
4. 6 in.

Examples—XXXIII.

1. 3 sq. ft. 30 sq. in.
2. 5 sq. ft. $77\frac{1}{2}$ sq. in.
3. 6 sq. ft. $69\frac{3}{7}$ sq. in.
4. 7 sq. ft. 48 sq. in.
5. 2 sq. ft. $139\frac{3}{7}$ sq. in.
6. 14 sq. ft. $92\frac{6}{7}$ sq. in.
7. 8 sq. ft. $7\frac{5}{7}$ sq. in.
8. 3 sq. ft. 118 sq. in.
9. 44 sq. ft. $28\frac{4}{7}$ sq. in.
10. 96 sq. ft. 36 sq. in.
11. 13 sq. ft. 64 sq. in.
12. 1 sq. ft. $138\frac{6}{7}$ sq. in.
13. 41 sq. ft. $130\frac{2}{7}$ sq. in.
14. 110 sq. ft. 132 sq. in.
15. 7 in.
16. 6 in.
17. 5 in.
18. 5 in.
19. 7 in.
20. 221 in.

Examination Questions—XXXIII.

1. 236·28 sq. ft. nearly.
2. $75\frac{3}{7}$ sq. in.
3. 221·269 sq. in.
4. 12·03 cub. ft.
5. £12.
6. £83 9*s.* 7·714*d.*

Examples—XXXIV.

1. 87 sq. in.
2. 6 sq. ft.
3. 6 sq. ft. 16 sq. in.
4. 85 sq. ft. 80 sq. in.
5. 9 sq. ft. 134 sq. in.
6. 112 sq. ft. 42 sq. in.
7. 37 sq. ft. 62 sq. in.
8. 60 sq. ft. 50 sq. in.
9. $267\frac{1}{7}$ sq. in.
10. 7 sq. ft. $13\frac{3}{7}$ sq. in.
11. £9 11*s.* $6\frac{73}{84}$*d.*
12. 505·17 . . . sq. in.

Examination Questions—XXXIV.

1. $73\frac{3}{4}$ sq. in. **2.** 1100 sq. ft. **4.** 1963·5 sq. in. nearly.
6. 346·62 sq. ft.

Examples—XXXV. A.

1. 616 sq. in. **2.** 68 sq. ft. 64 sq. in. **3.** 38 sq. ft. 72 sq. in.
4. 11 sq. yd. 4 sq. ft. $131\frac{1}{7}$ sq. in. **5.** 4 sq. ft. 40 sq. in.
6. 26 sq. ft. 106 sq. in. **7.** 38 sq. ft. 72 sq. in. **8.** 4·83 sq. ft. nearly.
9. 7 in. **10.** 4·58 in. **11.** 1 ft. 2 in.
12. 125 sq. ft. $102\frac{6}{7}$ sq. in. **13.** £4 12*s.* $4\frac{8}{21}d.$ **14.** £6 7*s.* $3\frac{3}{7}d.$
15. 2·04 in. **16.** $254\frac{4}{7}$ sq. in. **17.** £3 0*s.* $3\frac{9}{14}d.$
18. 4·81 sq. ft. **19.** 21 : 11. **20.** 215·6 sq. in.

Examples—XXXV. B.

1. 90 sq. in. **2.** 101·2 sq. in. **3.** 8·25 sq. in. **4.** 1 sq. ft. 66 sq. in.
5. 22·88 sq. in. **6.** 84·48 sq. in. **7.** 3 sq. ft. $105\frac{3}{7}$ sq. in.
8. 452 sq. ft. $82\frac{2}{7}$ sq. in. **9.** 1081 sq. ft. $20\frac{4}{7}$ sq. in.
10. 1631 sq. ft. $20\frac{4}{7}$ sq. in. **11.** 141·372 sq. in.
12. 40·231 sq. in. **13.** 3 in. **14.** $\frac{1}{12}$.

Examination Questions—XXXV.

1. 10*s.* $2\frac{58}{87}d.$ **2.** 4071·5006 sq. in. **3.** £95 14*s.*
4. 2,818,362$\frac{134}{693}$ sq. miles. **5.** 1386 sq. in. **6.** $47\frac{11}{36}$ cub. ft.
7. £466 4*s.* nearly. **8.** 128·47 . . . sq. ft. **9.** 5·38516 ft.
10. 18 in. **12.** 352 sq. ft. **13.** 6 ft.
14. 3 in. **15.** $\frac{3}{39652}$ of earth's surface. **16.** 2·07 in. nearly.
17. $1687\frac{5}{7}$ sq. ft. **18.** $745\frac{9}{14}$ sq. ft. **19.** 18π sq. in.
20. $181\frac{1}{35}$ sq. ft. **21.** $\frac{1}{4}$th of radius. **22.** $213\frac{3}{7}$ sq. ft.
23. $\frac{1}{82}$. **24.** 282,743 sq. ft. **25.** 151·3 sq. ft. nearly.
26. 10π sq. ft. ; 70π sq. ft. **27.** 379·77 sq. yds. nearly.

Examples—XXXVI.

1. 25 : 4. **2.** 64 : 49. **3.** 0·390625. **4.** 9 : 16.
5. 343 : 729. **6.** $\frac{1}{25}$. **7.** 1 : 3. **8.** 1·41421 ft.
9. 49 sq. in. **10.** 1 : 3 : 5.

Examination Questions—XXXVI.

1. 2·228 ft. ; 1·772 ft. **2.** 3·629 sq. ft. **3.** $3\sqrt{3}$: 1.

Miscellaneous Examples.

3. 562,500 sq. ft. ; 715,909$\frac{1}{11}$ sq. ft. **4.** $21\frac{3}{17}$ ft. **5.** 166 ft. $7\frac{1}{2}$ in., or 85 ft.
6. 56 in. **8.** 7·483 sq. in. **9.** 1·73205 yds.
11. 3·2083 cub. ft. ; Rs.19 4 annas. **12.** $19\frac{3}{6}$ ft.
13. $312\frac{1}{2}$ cub. ft. **14.** 750 sq. yds. **16.** 6 ft. ; 47·74°.
17. 240 tons. **19.** Less by 0·8 . . . sq. ft. **20.** 1924 sq. in.
21. $50\frac{10}{11}$ in. **22.** 269 ft. ; 323 ft. ; 431 ft. **23.** 16·8253 sq. ft.
24. $980\frac{4}{7}$ lbs. **25.** $9\frac{5}{8}$ sq. in. **26.** $15\frac{10}{11}$ in.
27. 100 yds. ; 7857$\frac{1}{7}$ sq. yds. **28.** 64.

29. 1318·0349009 sq. ft. **30.** 1924·50 . . . sq. ft. **31.** 29·8 ft.
32. 949 cub. ft. nearly. **33.** $53\frac{1}{3}$; 442·7 nearly. **34.** $638\frac{23}{48}$ sq. ft.
35. 28·62426 acres. **36.** 125 yds. **37.** 7 ft.
38. 2574 sq. ft.; $62\frac{1}{2}$ ft.; $41\frac{231}{625}$ ft. **39.** 2·3776 : 3·6327.
40. 1 : 4. **41.** 48·6 . . . galls. **42.** $31\frac{1}{4}$ ozs.
43. 12·40; 10·74; 12·40. **44.** 66·25 yds. **45.** 15,246 sq. ft.
46. 340,460 sq. ft. nearly. **47.** 207·846 ft.
48. No; because each of the MN parallelograms is not necessarily the corresponding unit of area. **49.** 93·5 . . . sq. in.
50. 31·0304 . . . yds. **51.** 259 shot. **52.** 193·4 ft.
53. $\pi : 2$. **54.** 45·033 . . . ft. **55.** 8 ac. 0 ro. 21·355 po.
56. 1 ac. **57.** $\frac{p^2}{4\pi}$ sq. units. **58.** £47 2*s.* 5·724*d.*
59. 129·903 . . . sq. ft. **61.** 3·83296 ac. **62.** 1963·5 sq. in.
63. 8246·7 cub. in. **64.** Rs.210 8 annas nearly.
66. 1524·204 . . . yds. or 762·102 . . . yds. **67.** 498·83 . . . sq. ft.
68. 594·18 . . . sq. ft. **69.** 409·449 . . . sq. in.; 50·64 . . . in.
71. 3·3540 . . . ft. **72.** 1931·37 sq. ft.; 1831·79 sq. ft.
73. $3 : 2\sqrt{3}$. **74.** $4\frac{1}{2}$ ft. and 9 ft. **75.** 1400 sq. ft.
76. 22 ft.; 23° 26′ 20″. **77.** 82·7 sq. in. **78.** 2·35 ac.
80. 25·033 ft. nearly. **81.** 3·099 ft.; 22 ft.
82. 8π cub. ft.; 6π cub. ft. **83.** 10,164 sq. ft. **84.** 706·5.
85. $\frac{d(l_1+l_2)}{2}$; 43·63 . . . sq. ft. **86.** 3·41421 ft. **87.** 69·2 ft. nearly.
88. 18,775 sq. yds. **90.** Rs.26 13 annas 2·3 pies.
91. 6 in.; 113·0976 cub. in. **92.** 13,330 sq. ft. **93.** 424·26 . . . sq. ch.
94. 886·81 sq. ft. **95.** $141\frac{2}{3}$. **96.** $2619\frac{9}{14}$ sq. ft.
97. 8·496 ac. **98.** 25 in. **99.** 9960 sq. ft.
100. $204\frac{17}{32}$ cub. ft.; 11·17 ft. **101.** 201 nails. **102.** 30 ft. 6 in.
103. $117\frac{1}{3}$ cub. ft. **104.** 6·03416 ft. **105.** 839·553 . . . sq. ft.
106. 183·71 . . . ft. **107.** 187 cub. ft. 2 primes 2 thirds.
108. 836,750 sq. lks. **109.** 10·003 . . . yds.
110. 14,322 sq. lks. **111.** 72·14 lbs.
112. 4400 cub. ft. **113.** 3540 sq. yds. **114.** 2·82 . . .
115. 4·669 . . . ft. **116.** 10·504 in. **117.** 3 ft. 9 in.
118. 3·5805 ac. **119.** 46·28 cub. ft. **120.** 373 lbs. nearly.
121. 9 in. **122.** 600 sq. yds. **123.** 199·102 . . . sq. ch.
124. £96 6*s.* **125.** 48 ft. **126.** $1026\frac{2}{3}$; 3080; $5133\frac{1}{3}$.
127. 764·53. **128.** 5·42 in. nearly. **129.** 0·1545 ozs.
130. 3147·18 cub. ft. **131.** 3274. **132.** 14,400 sq. ft.
133. 30 ft.; 55 ft. **134.** 5·058 . . . ft. **135.** 6·11 in. nearly.
136. 6·336 in. to 1 mile. **137.** 246·5 sq. in. nearly. **138.** 17·6715 cub. ft.
139. 195·8264 cub. in. **140.** 8·32 ft. **141.** 455·498 . . . sq. yds.
142. 6813·52 sq. ft. **143.** 30,780.
145. Fence 17·008 ch. distant from smaller boundary. **146.** 65·45 cub. ft.
147. 204 cub. ft. 6′ 4″ 6‴. **148.** £11 0*s.* 6*d.*; £7 12*s.* 3*d.* **149.** $111\frac{9}{16}$ cub. in.
150. 220 shot. **151.** $\frac{\pi h}{12}(d^2 + dd' + d'^2)$. **152.** 5194·112 cub. ft.
153. 60 ft. **154.** 0·35182 sq. ft. **155.** 1 : 660.
156. 102,275 cub. ft. nearly. **157.** 6·933 ft.; 1·802 ft.; 1·264 ft.
158. 832,000 cub. ft. **159.** 27·65 lbs.
161. 46·08 sq. ft.; 160·44 sq. ft. **162.** 90. **163.** 4 in.
164. 1·41421 ft. **165.** £5111 11*s.* 0*d.* nearly. **166.** 0·684 in.
167. 168 sq. in. **168.** $391\frac{1}{2}$ sq. in. **169.** 5373 cub. in. nearly.
170. $233\frac{1}{3}$ cub. ft. **171.** 1·18215 in.

172. 2 ft. 11 in. by 1 ft. 6 in. **173.** $r^2\left(\sqrt{3}-\frac{\pi}{2}\right)$.

174. $54\frac{6}{11}$ min. past 10; $21\frac{8}{11}$ min. past 10; $38\frac{2}{11}$ and $5\frac{5}{11}$ min. past 10; $16\frac{4}{11}$ min. past 10. **175.** 3·114 . . . ft.

176. 34·8 in. nearly. **177.** $1329\frac{3}{7}$ cub. in.; $933\frac{3}{7}$ sq. in.

178. 3·82 cub. ft. **179.** 1728 marbles. **180.** 21 yds. nearly.

181. 3 : 1. **182.** 3·833 cwt.; £979 10*s.* 6*d.* nearly.

183. Rs.24,095 nearly. **184.** 50 ft.

185. 9·175 . . . in.; 11·957 . . . in.; 28·867 . . . in. **186.** $4\frac{1}{8}$ cub. ft.

187. 27·734 . . . in.; 7·208 . . . in.; 5·056 . . . in. **188.** 15,050 cub. yds.

189. 6196 cub. in. **190.** 6·196 . . . in. **191.** 59,136.

192. 14·3 ozs. nearly. **193.** 280 yds. **194.** 8028 cub. ft. nearly.

196. 42·26 . . . ft. **197.** Length, 24 ft.; breadth, $16\frac{2}{3}$ ft.

198. Rs.200. **199.** 75·3984 sq. ft. **200.** 103,119 galls. nearly.

201. 11 cwt. 0 qrs. $26\frac{1}{3}$ lbs. **202.** 3·8 . . . ft.

203. 1340 cub. ft. nearly. **204.** 91 ft. 8 in.

205. Height, 10 ft.; length, 21 ft.; breadth, $10\frac{1}{2}$ ft. **206.** 5·19615 in.

207. $\frac{\pi-2}{\pi}$. **208.** 11·75 cub. in. **209.** 744,000 cub. ft.

210. 433·012 cub. in. **211.** $7657\frac{1}{7}$ cub. ft. **212.** 3703·2 cub. ft.

213. 86·602 yds. **215.** 20·3 sq. ft. nearly; 686·4 sq. ft. nearly.

216. $458\frac{1}{3}$ cub. in.; $458\frac{1}{3}$ cub. in.; $851\frac{4}{7}$ cub. in. **217.** $19\frac{17}{27}$ cub. ft.

218. 2485 shot. **219.** $283\frac{1}{4}$ cub. ft.; $247\frac{25}{28}$ cub. ft.

220. 588 cub. in. **221.** 1060·8 cub. ft. nearly; Rs.371 4 annas nearly.

222. 1·65 . . . ft.; 3·59 . . . ft. **223.** Rs.988 nearly.

224. Rs.239 13 annas 10 pies nearly. **225.** 4·898 in.

226. $65{,}066\frac{2}{3}$ cub. ft. **227.** 92 sq. ft. 96 sq. in nearly.

228. 9·4248 cub. ft. **229.** 3·577 . . . ft. from smaller end; 19π cub. ft.

230. $\frac{1}{4226}$ of the earth's surface. **231.** 1058·6 yds.

232. 20,000 ft. **233.** 0·519 in. nearly. **234.** 3·96 in.

235. 4·87 sq. ft.; 0·4249 cub. ft. **236.** 740 miles an hour.

237. 143 pints. **238.** 25·455 in. **239.** 304 cub. in.

240. $161\frac{85}{189}$ cub. ft.; $176\frac{22}{27}$ cub. ft.; $192\frac{166}{189}$ cub. ft. **241.** $\frac{a^2}{2}(9+\sqrt{3})$.

242. Rs.50 9 annas. **243.** $r^2 \times n \tan\frac{180^\circ}{n}$. **244.** 5804 cub. ft. nearly.

245. 6550 cub. ft. nearly. **246.** 30·5 lbs. nearly.

247. 150·857 sq. in.; 387·09 cub. in.

249. 2·00 . . . in.; 1·61 . . . in.; 1·37 . . . in. **250.** 88·106 cub. in.

251. $335\frac{5}{21}$ cub. in.; $335\frac{5}{21}$ cub. in.; $737\frac{11}{21}$ cub. in.; $737\frac{11}{21}$ cub. in.

252. 1201·6 cub. ft.; 376·9 cub. ft. **253.** 12π sq. in.

254. 4·49 in. **255.** Rs.177 15 annas 8 pies nearly.

256. $2\frac{3}{4}$ in. nearly. **257.** $67{,}598\frac{2}{3}$ cub. yds. **258.** 7·542 cub. in.

259. 20,820,690 sq. miles nearly. **261.** $73\frac{1}{4}$ sq. in.

263. 15·8 sq. in. **264.** 432 cub. in. **265.** £23 14*s.* 1·96*d.*

266. 13·867 . . . ft.; 3·604 . . . ft.; 2·528 . . . ft. **267.** 45 in.

268. 5·76 in. **269.** 21 sq. ft. 128 sq. in. nearly.

270. 7·029 ft. **271.** 3037 cub. ft. Rs.1063 nearly.

272. 11·696 in.; 3·04 in.; 2·132 in.; 1·698 in.; 1·434 in.

273. 0·16 in. **274.** 1540 balls; 99 in. nearly.

275. 88·5 cub. in. nearly. **277.** 22·654 sq. yds. **278.** 297,666 galls. nearly

279. Rs.695·69. **280.** 51·056 cub. ft. **281.** 63 galls. nearly.

282. 3·09 . . . cub. in. **283.** 18·1649 . . . ft. **284.** $157\frac{1}{7}$ ft.

285. 2 qrs. 2 lbs. 14 oz. nearly. **286.** 140,274·314 cub. ft.

287 $\frac{1}{16}$. **288.** $(6r+a\sqrt{3}) : (6r+2a\sqrt{3})$.

289. $h \times \sqrt[3]{1 - \frac{4r^3}{a^2h}}$. **290.** $\frac{2\sqrt{2}}{3} \cdot a^2h$. **292.** 0·018 . . . ozs.

293. 32 sq. ft. **294.** 1·35 ft. nearly. **295.** 0·654 cub. in.

296. 1·9 . . . ft. ; 3·5 . . . ft.

297. Height, 10·5 ft. ; side of base, 5·6124 ft. ; cost, Rs.76 3 annas 11 pies.

298. 355·5 cub. in. nearly. **299.** $70\frac{3}{4}$ yds. nearly.

300. $7903\frac{467}{686}$ cub. in. **301.** 6588 cub. ft.

302. 0·216 cub. ft. **303.** $17\frac{13}{84}$ cub. in.

Additional Examination Questions.

XX.

20. 103·923 yds. ; Rs.1064 nearly. **21.** $10\frac{10}{11}$ miles per hour.

23. $754\frac{13}{18}$ sq. ft. **24.** 11,680 sovs.

XXII.

59. 14,520 tanks. **60.** 96 mins. **61.** 4602 lbs. nearly.

62. 51,606 galls. nearly. **63.** $15,639\frac{3}{11}$ yds.

64. $15\frac{219}{784}$ cub. in. ; $19\frac{311}{3136}$ cub. in.

XXIII.

39. 15·705 galls.

XXIV.

13. 3240 cub. in.

XXV.

5. $112\frac{53}{56}$ cub. ft.

XXVI.

67. $13,621\frac{1}{3}$ cub. ft.

XXVIII.

22. $2413\frac{3}{7}$ cub. in. **23.** 1980 cub. in.

XXIX.

15. 81 in. ; 108 in. **16.** $4\frac{5}{47}$ cub. in. ; 3·1748 in.

XXX.

26. 8·2915 in. ; 237·1 . . . sq. in. **27.** $34\frac{3}{4}$ sq. ft.

28. 10 ft. ; 7 ft. ; 8 ft. **29.** 46·7 sq. ft. nearly.

XXXI.

8. $183\frac{69}{198}$ sq. in. **9.** Rs. 800.

XXXIII.

7. $\frac{1}{2}Cs$. **8.** $83\frac{17}{21}$ ft. **9.** $39\frac{1}{9}$ sq. yds. nearly.

XXXV.

28. $21\frac{19}{81}$ per cent. **29.** $\frac{3}{16}$. **30.** Rs. 18,072 12 ans.

31. $51\frac{1}{14}$ sq. ft. nearly.

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